LUCIANO SOUZA

NEW TRIGONOMETRIC CLASSES OF PROBABILISTIC DISTRIBUTIONS

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PRÓ-REITORIA DE PESQUISA E PÓS-GRADUAÇÃO

DOUTORADO EM BIOMETRIA E ESTATÍSTICA APLICADA

NEW TRIGONOMETRIC CLASSES OF PROBABILISTIC DISTRIBUTIONS

Tese apresentada ao Programa de Pós-Graduação em Biometria e Estatística Aplicada, Universidade Federal Rural de Pernambuco, para obtenção do título de Doutor em Biometria e Estatística Aplicada, Area de Concentração: Estatística Aplicada.

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LUCIANO SOUZA

Tese julgada adequada para a obtenção do título de Doutor em Biometria e Estatística Aplicada, defendida e aprovada por unanimidade em 13/11/2015 pela banca examinadora.

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DEDICATÓRIA

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"Embora ninguém possa voltar atrás e fazer um novo começo, qualquer um pode começar agora e fazer um novo fim".
Chico Xavier.

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Resumo

Nesta tese apresentamos e investigamos quatro novas classes trigonométricas de distribuições probabilísticas. As classes seno, cosseno, tangente e secante. Para cada uma das novas classes foi criada uma nova distribuição. Estas quatro novas distribuições foram usadas na modelagem de dados reais. Obtivemos uma simulação numérica, usando como base a distribuição exponencial, para se comparar os vicios (bias) e verificamos que, a medida que aumentamos o tamanho da amostra, o bias tende a zero. Alguns resultados numéricos para ver estimativas de máxima verossimilhança e os resultados para amostras finitas foram obtidos. Várias propriedades das classes e as suas distribuições foram obtidos. Obtemos as expansões, as estimativas de máxima verossimilhança, informações de Fisher, os quatro primeiros momentos, média, variância, assimetria e curtose, a função geradora de momentos e a entropia Rényi. Mostramos que todas as distribuições têm proporcionado bons ajustes quando aplicadas a dados reais, em comparação com outros modelos. Na comparação dos modelos foram utilizados: o Akaike Information Criterion (AIC), o Akaike Information Criterion Corrigido (CAIC), a informação Bayesian Criterion (BIC), o critério de informação Hannan Quinn (HQIC) e duas das principais estatísticas também foram utilizadas: Cramer -von Mises e Anderson-Darling. Por fim, apresentamos os resultados da análise e comparação dos resultados, e orientações para trabalhos futuros.

Palavras-chaves: classes trigonométricas de distribuições probabilísticas, funções univariadas.

Abstract

In this thesis, four new probabilistic distribution classes are presented and investigated: sine, cosine, tangent and secant. For each of which a new kind of distribution was created, which were used for modelling real life data.By having an exponential distribution to compare the biases, a numerical simulation was obtained, making it possible to verify that the bias tends to zero as the sample size is increased. In addition to that, some numerical results for checking maximum likelihood estimates, as well as the results for finite samples, were obtained, just as much as several class properties and their respective distributions were also obtained, along with the expansions, maximum likelihood estimates, Fisher information, the first four moments, average, variance, skewness, and kurtosis, the generating function of moments and Renyi's entropy. It was evidenced that all distributions have shown good fit when applied to real life data, when in comparison to other models. In order to compare the models, the Akaike Information Criterion (AIC), the Corrected Akaike Information Criterion (CAIC), the Bayesian Information Criterion (BIC), the Hannan Quinn Information Criterion (HQIC) were used, along with two other main statistic sources: Cramer-Von Mises and Anderson-Darling. As a final step, the results of the analyses and the comparison of the results are brought up, as well as a few directions for future works.

Keywords: trigonometric classes of probability distributions, univariate functions.

CHAPTER 1

Introduction

This thesis consists of four independent articles, in which, viewed as separate chapters, a new class of trigonometric distributions is proposed. Each of these chapters can be read independently of each other, since each one is self-contained. As each chapter contains a thorough introduction to the state of the art and motivations, this introduction just contains a general, brief review of each chapter.

The initial motivation for the origin of these new classes was modelling a real problem known as the problem of Kiama Blowhole. Thus the main objective of this thesis was the search for flexibility of modelling even in the presence of small variations in the sample. This led to the proposal for the creation of these new classes of trigonometric probability distributions.

According to Johnson et al. [6] in recent years the statistical area has become increasingly promising for modelling and prediction. In the literature studies of probability distributions and their applications to real data in various fields of human knowledge abound. The recent growth of new distribution proposals has widened the field of applications. Most of these new distributions are far better explanations of the phenomena that were once known, and are able to give better predictions.

According to Lee et al. [8], univariate distributions are derived in three ways: method of differential equations, processing methods and the method of quantification. However, there is still a great need for more flexible models in several areas such as Genetics, Medicine, Agronomy, Engineering, Economics, survival analysis among others. We need models that are able to provide a better explanation of the phenomenon studied, so we can have a better understanding of the factors involved, as well as in the development of better predictions.

According to Ramos [10] the data may show a high degree of skewness and kurtosis as well as new models need to have a larger number of parameters which can provide greater control over those measures. Most statistical models proposed in the literature have a large number of parameters in an attempt to make the model more flexible. It is desirable however to develop models that have a small number of parameters and at the same time with a large degree of flexibility for modelling the data.

In each chapter, we propose a new distribution, useful to model risk and survival phenomena. We developed the Rényi entropy for the new models. In the literature, there are a few researchers who decided to seek new distributions using trigonometric distributions. Burrows [2] was one of the first to suggest a sine distribution; Gilbert [4] proposed a distribution that bears his name (Gilbert sine distribution) and, more recently, Nadarajah and Kotz [9], proposed the trigonometric beta distribution Al-Faris et al. [1] proposed the square sine.

In Chapter 2, we present a brief review on survival analysis and some classical and trigonometric distribution as well as some other special classes.

In Chapter 3, we propose our first new class of trigonometric distributions, called the Sine Class. The inverse Weibull model has the capability of failure rates, this new class has the Sin inverse Weibull distribution as a special case. The inverse Weibull distribution is very useful in modeling failure rates, so the new distribution also proved to be very flexible in the modeling of these type of data. The model has greater applicability to problems in the biological area, since most of these problems have unimodal failure rate.

Some authors introduce variations of the inverse Weibull distribution such as Gusmão et al. [5] who proposed a generalization of the inverse Weibull distribution. The generalized model, also known as reversed Weibull, has the inverse Weibull distributionas a particular case, and is thus more flexible.

In Chapter 4, we propose another new trigonometric distribution class, the cosine class, which has the Weibull distribution as a particular case. The Weibull distribution is widely known in the literature by its great flexibility in modelling survival phenomena and the new distribution proved to be flexible in similar situations.

In Chapter 5, we propose another new class of trigonometric distribution, the tangent class, which has the Tan-BXII distribution as a particular case. The Burr XII distribution has the Gaussian, log-normal and gamma distributions, among others, as particular cases and the development of this new distribution gives various forms for the risk function.

In Chapter 6, we propose our last new trigonometric distribution class, the secant class, which has the Sec-KumW as a particular case. The Kumaraswamy distribution was first applied to hydrology data, but its applications have spread to other fields in recent years. Some newer models were applied to lifetime data from patients with bladder cancer and the results were quite consistent. The new distribution Sec-KumW was designed with the purpose of modelling data on the hydrology and flow of rivers and more recently a proposal to model survival data.

In Chapter 7, to compare the four classes, we conducted a numerical simulation with different sample sizes (50, 100, 200 and 1000). For each sample, 5 thousands replicas of Monte Carlo were tested. We obtained the estimates by maximum likelihood, variance, EQM and error. We note that the EQM of the distribution Sec-Exp had lower values, the measure that increased the sample size.

In Chapter 8, we have the conclusion, considerations and suggestions for future works.

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CHAPTER 2

Literature Review

In this chapter, we will present the Weibull, Reverse Weibull, Burr XII and Kumaraswamy-Weibull distributions, the maximum likelihood estimation and moments methods, as well as some criteria for model selection such as AIC, AICC, BIC and HQIC. Two tests which will be largely used in this thesis are the Anderson-Darling and Crammer-von Misses tests. Occasionally, other tests will be used and their definitions and motivations explained when needed.

2.1 Survival analysis

Survival analysis is a statistical technique that deals with Colosimo and Giolo [15] censored data, which, by its turn, are partial observations of the response variable. Censored data are those for which not all the values of observations are known. In other words, they are partially known but not truncated in a sense that the known values are exact. A good example is an interrupted patient follow-up, either because the patient moved to another city or because s/he died of unrelated cause (or any other cause not related to the observation);

the data analysis study could, then, be accomplished by using survival data analysis methods. Another common application is when the failure time of an object, e.g. a light bub, is summed up and the failure time is greater than the observed one. In the survival analysis model, the variable of interest is the time of occurrence of an event of interest, defined as "hazard time". Hazard is a process that can be defined as the time until breakage of equipment or the time until the death of a biological organism as well as the cure or recurrence of a disease Lawless [38].

2.1.1 Survival function and Hazard function

The *survival function*, denoted by S, is defined as

$$S(t) \quad = \quad P(T \geq t)$$

The *lifetime distribution function*, denoted by F, is defined as:

$$F(t) \quad = \quad \Pr(T \leq t),$$

thus, we have

$$\begin{aligned} F(t) &= 1 - S(t) \\ S(t) &= 1 - F(t)\cdot \end{aligned} \tag{2.1}$$

The hazard function is defined as the following expression:

$$h(t) = \lim_{\Delta t \to 0} P\frac{(t \leq T < t + \Delta t | T \geq t)}{\Delta t}.$$

2.1.2 Types of censorship

We define type I censorship when the study is expected to be completed in a period of time already set, regardless of the occurrence of the event of interest. The opposite occurs with type II censorship, where data acquisition is stopped after a set number of occurrences of the event of interest. From now on, well call the event of interest *failure*. We define the random censorship when the patient or the equipment is removed from the study without the fault having been observed. In this example, quite commonly seen in the medical field, a patient is likely to die, for reasons without relation to the study of which he participates. The random censoring is based on two random variables T and C. The former is defined as representing the random failure time of a patient or equipment, and a random availability time is represented by the latter. It is important to remind that T and C are independent Colosimo and Giolo [15]. We are interested in watching a particular information where $t = min(T, C)$

$$\delta = \begin{cases} 1, & \text{se} \quad T \leq C \\ 0, & \text{se} \quad T > C, \end{cases}$$

in that $\delta = 0$ indicates censorship - failure didn't happen during the allotted amount of time - and $\delta = 1$ indicates an observed failure.

2.2 Distributions

2.2.1 Weibull distribution

The Weibull (W) distribution, proposed by Weibull [74], has been used in many different fields with many applications, for example, Murthy et al. [45]. The Weibull distribution is widely known as one of the most applied probability distribution in survival analysis and reliability theory. Its failure rate may be increasing, decreasing or constant. Since it does not show non-monotonic or unimodal forms, it can not be used for modeling survival data that have failure rates with bathtub form, such those of human mortality and equipment life-cycle. Since 1958, the Weibull distribution has been modified by many researchers to allow non-monotonic risk function. Researchers have developed several extensions and modified forms of the Weibull distribution with a number of parameters ranging from 2 to 5. The two-parameter Weibull proposed by Bebbington et al. [6] has a failure rate that can be increasing, decreasing or have a bathtub format. Zhang and Xie [75] studied the characteristics and shapes of the Weibull distribution applications and the truncated Weibull has failure rate in bathtub shape. A model with three parameters was proposed by Marshall and Olkin [40], they called it the extended Weibull distribution. Another model with three parameters, called Weibull exponentiated, was introduced by Mudholkar and Srivastava [43]. The Weibull probability density function (pdf) is given by Eq. (2.2)

$$g(x) = \alpha\lambda^{\alpha}x^{\alpha-1}\exp\left[-(\lambda x)^{\alpha}\right], x > 0, \tag{2.2}$$

where $\alpha > 0$ and $\lambda > 0$. The cumulative distribution function (cdf) is given by Eq. (2.3)

$$G(x) = 1 - \exp\left[-(\lambda x)^{\alpha}\right], x > 0\cdot \tag{2.3}$$

2.2.2 Inverse Weibull distribution

The Inverse Weibull (IW) distribution has received much attention in the literature. In Keller and Kamath [35], the use of the IW distribution for describing the degeneration phenomena of mechanical components is introduced. The IW distribution also provides a good fit to several kinds of data such as the breakdown times of an insulating fluid, subject to the action of constant tension Nelson [53]. This distribution is also called Reverse Weibull distribution Simiu and Heckert [63], Additional Weibull distribution Drapella [22] and the Reciprocal Weibull distribution Mudholkar and Kollia [42]. Some statistical order properties of the IW distribution are derived in Abdur and Menon [1]. In Calabria and Pulcini [13], the maximum likelihood and least square estimation of parameters are discussed. If the random variable Y has Weibull distribution, then $X = \dfrac{1}{Y}$ has IW distribution, and its cumulative distribution function (cdf) is given by

$$G(x) = \exp(-\alpha x^{-\theta}), x > 0 \cdot$$

Where $\alpha > 0$ and $\theta > 0$ are scale and shape parameters, respectively. The probability density function (pdf) is given by

$$g(x) = \alpha\theta x^{-\theta-1}\exp(-\alpha x^{-\theta}), x > 0 \cdot$$

Its hazard rate function (hrf) is given by

$$h(x) = \alpha\theta x^{-\theta-1}, x > 0 \cdot$$

The hrf of the IW distribution is unimodal in shape. We obtain some possible shapes of the density for different parameter values and some possible shapes of the hazard rate function.

2.2.3 Burr XII distribution

The Burr XII (B) distribution was proposed by Burr [11]. The flexibility of Burr XII was investigated by Hatke [32], Burr [12], Rodrigues [61] and Tadikamalla [70]. In different directions Takhasi [71] was the first to report that the Burr XII density function with three parameters can be obtained by the composition of the Weibull density function with the density function of the Gamma. The Burr XII cdf is given by

$$G(x) \;=\; 1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k} \;, x > 0 \;,$$

where $c > 0$, $s > 0$ and $k > 0$ are parameters, respectively. The probability density function (pdf) is given by

$$g(x) \;=\; x^{c-1} c k s^{-c} \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k-1} \;, x > 0 \;\cdot$$

Burr XII and their mutual distributions Burr III have been used in a wide variety of applications, such as actuarial science, according Embrechts et al. [23], in a biologic essay, according Drane et al. [21], and economy, as MacDonald and Richards [39], and Morrisson and Schmittlein [41].

2.2.4 Kumaraswamy-Weibull distribution

Kumaraswamy [5] questioned the flexibility of the beta distribution as not so efficient in real data applications such as from rain, daily stream flow etc. Jones [34] stated that the beta distribution is flexible, but not that much. The Kumaraswamy (Kum) is a preferable distribution to the beta distribution, besides being more flexible. A Kum with a number of parameters can produce a large amount of functional forms. Cordeiro and Ortega [17], and Kumaraswamy [5], proposed a distribution with only two parameters and the interval (0; 1) then called it Kumaraswamy distribution. According Cordeiro and Ortega [17] the Kumaraswamy distribution is not well known among statisticians, although some have begun to work in this distribution. Codeiro and Ortega [17] recently developed a Kumaraswamy Weibull distribution with 8 sub models and a modified Kumaraswamy Weibull distribution Cordeiro and Ortega [18]. Ramos [57] proposed a new family of continuous distributions called Kumaraswamy-G Poisson with three parameters. The cumulative density function distribution (cdf) and probability density function (pdf) can be seen below:

$$G(x) \;=\; 1-\left[1-(1-\exp[-(\lambda x)^{c}])^{a}\right]^{b}\;, x>0,$$

and

$$g(x) = abc\lambda^{c}x^{(c-1)}\exp[-(\lambda x)^{c}]\{1-\exp[-(\lambda x)^{c}]\}^{(a-1)}$$
$$\times\{1-[1-\exp(-\lambda x)^{c}]^{a}\}^{(b-1)}\;, x>0,$$

with $a>0$, $b>0$, $c>0$ and $\lambda>0$, respectively.

2.3 Trigonometric distributions

2.3.1 Sine distribution

In the last century, the astronomer Gilbert [27] proposed the sine distribution. The creation of the model arose from the need for calculating the angles of impact in the craters of the moon caused by meteorite, as follows

$$f(x) = \sin 2x, 0 \leq x \leq \pi/2\cdot$$

2.3.2 Cosine distribution

Raab [56] proposed using a cosine distribution to approximate the normal distribution, as seen below

$$f(x) = \frac{1}{2\pi}(1+\cos x), -\pi \leq x \leq \pi\cdot$$

The fact of the trigonometric function has several features and the fact that the cosine be platykurtic (compressed) as compared to the normal distribution, when your fit is not poor Raab [56].

2.4 The beta trigonometric

The beta trigonometric, proposed by Nadarajah and Gupta [46], can be used to model economic data, as follows

$$f(x) = Cx^{\upsilon-1}(1-x)^{\mu-1}\cos(ax),$$

for $0 < x < 1$, $\upsilon > 0$, $\mu > 0$ and $0 \leq a \leq \pi/2$, where the constant C is given by

$$\frac{1}{C} = \frac{1}{2}B(\upsilon,\mu)_1F_2\left(\upsilon;\frac{1}{2},\upsilon+\mu;-\frac{a^2}{4}\right).$$

2.5 Circular distributions

The circular distribution has been used in many applications such as insect lifetime and is widely applied in geology Lark [37]. The measures circulars are recorded in phenomena with comments signals (direction of movement of an animal after a given stimulus, wind direction, etc.) as well as in periodic phenomena (arrival time a patient from a hospital, visit of insects on flowers) Izbick and Esteves [33]. The Von Mises distribution was proposed by Von Mises [73]. A circular distribution proposed in Fernadez [26], Qin [69] models data on wind speed. Struknov [68] studied the properties and shapes of the Von Mises distribution.

Definition 2.5.1. *Is ϕ a random variable such that their density is given by*

$$f(\phi) = \frac{1}{2\pi}I_{[0,2\pi]}(\phi) \in (0;2\pi),$$

where ϕ has uniform distribution in the $(0,2\pi)$, $\phi \sim (0,2\pi)$.

Definition 2.5.2. *Is ϕ a random variable such that their density is given by*

$$f_\theta(\theta;\theta_0,k) = \frac{1}{2\pi I_0(k)}\exp\left\{k\cos(\theta-\theta_0)\right\}, k \leq 0,\ \theta;\ \theta_0 \in U(0;2\pi),$$

and $I_v(k)$ is a modified Bessel function. We say that the parameters ϕ given θ and κ have distribution of Von Mises, $\phi|\theta,\kappa \sim VM(\theta\kappa)$.

2.6 Class of distributions

There are in the statistical literature many probability distribution classes. We present some of these in this work.

2.6.1 Classes of exponentiated distributions

The exponentiated class was defined by Mudholkar and Kollia [42]. Using a $G(x)$ a cumulative distribution function as baseline, the distribution is defined as

$$G_\alpha(x) = G^\alpha(x),$$

with power parameter $\alpha > 0$. The corresponding pdf is expressed by

$$g_\alpha(x) = \alpha g(x) G^{\alpha-1}(x)\cdot$$

Several researchers developed new properties and new classes in search of better models such as Gupta and Kundu [29], who developed the generalized exponential distribution. Nadarajah and Kotz [47], Nadarajah and Kotz [49] and Rao et al. [59] introducing exponentiated Fréchet, exponentiated Gumbel and exponentiated log-logistic distributions. Ahuja and Nash [2] introducing exponentiated exponential distribution, Mudholkar and Srivastava [44] introducing the exponentiated Weibull family, Gupta et al. [28] introducing the exponentiated Pareto distribution and Surles and Padgett [65], [66] and [67] introducing the exponentiated Rayleigh distribution.

2.6.2 Beta distribution class

One of the most important classes is the Beta class. According with Eugene et al. [24], the cumulative, density function of the beta distribution classes is given by

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw,$$

where $a > 0$ and $b > 0$ are shape parameters and $B(a,b)$ is beta function defined by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\cdot$$

Nadarajah and Gupta [46] obtained the Beta Fréchet, Nadarajah and Kotz [51] proposed the Beta exponential distribution, Nadarajah and Kotz [48] obtained the Beta Gumbel distribution and Faymoye et al. [25] introduced the Beta Weibull distribution. More recently, Hanook et al. [31] proposed the Beta inverse Weibull distribution, Castellars et al. [14] proposed the Beta lognormal distribution, Shittu and Adepoju [64] obtained the Beta Nakagami distribution and Baharith et al. [5] proposed the Beta generalized inverse Weibull distribution.

2.6.3 Classes of generalized Kumaraswamy distributions

Cordeiro and Castro [16] defined the Kumaraswamy-G class, based on the generalization of Kumaraswamy distribution proposed by Kumaraswamy [36], whose cdf is given by

$$F(x) = 1 - (1 - x^a)^b\,,$$

with $a > 0$, $b > 0$ and $0 < x < 1$. Where the Kumaraswamy-G class comes as proposed below

$$F(x) = 1 - \{1 - G(x)^a\}^b\,, 0\ <\ G(x)\ <\ 1\cdot$$

$G(x)$ is baseline, $0 < G(x) < 1$ and $a >$, $b > 0$ are shape parameters. More recently, Ramos [57] proposed the Kumaraswamy-G Poisson Family of distribution, Cordeiro et al. [19] introduced the Kumaraswamy Gumbel distribution, Bourguignon et al. [8] proposed the Kumaraswamy Pareto distribution and Nadarajah and Eljabri [52] to define the new model the Kumaraswamy generalized Pareto distribution.

2.6.4 Gamma distribution class

The gamma distribution class was developed by Zografos and Balakrishnan [76]. Its cdf is given by

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{-\log[S(x)]} t^\alpha \exp\{-t\}\, dt, \tag{2.4}$$

where $\alpha > 0$ and $S(x) = 1 - G(x)$ is survival function and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is gamma function. More recently, Ristic and Balakrishnan [60] proposed the exponential exponentiated gamma and Pinho et al. [54] proposes a generalization called gamma weibull exponentiated and studied their properties. Ramos et al. [58] proposed a generalization of the template Zografos and Balakrishnan [76] by proposing a new model called Zografos and Balakrishnan-G where studied the properties and underwent applications and Torabi and Montazeri [72] defined the class gamma $G(x)/(1-G(x))$ by

$$F(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{G(x)/(1-G(x))} t^{\alpha-1} \exp\{-\beta/t\} \cdot$$

Brito [7] proposed the gamma $(1-G(x))/G(x)$ distribution class. The new following nomenclature type tends to facilitate the standardization of new classes currently generated:

$$H_G(x) = \int_{(1-G(x))/G(x)}^{+\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} \exp\{-\beta t\}\, dt$$

2.6.5 Generalized Weibull distributions class

Barros [4] proposed four new classes of Weibull distributions. She also made use of the new nomenclature proposed by Brito [7]. The new classes are defined as

- The Weibull class $(\delta(1-G(x)), \delta - \log(1-G(x)))$

$$H_G(x) = \int_{\delta(1-G(x))}^{\delta-\log(1-G(x))} \lambda\theta(\lambda t)^{\theta-1} e^{-(\lambda t)^{\theta}} dt, \delta \geq 0, x > 0.$$

- The Weibull class $(\delta G(x), \delta - \log(G(x))$

$$H_G(x) = 1 - \int_{\delta G(x)}^{\delta-\log(G(x))} \lambda\theta(\lambda t)^{\theta-1} e^{-(\lambda t)^{\theta}} dt, \delta \geq 0, x > 0 \cdot$$

- The Weibull class $\left(\delta(1-G(x)), \dfrac{G(x)}{1-G(x)}\right)$

$$H_G(x) = \int_{\delta(1-G(x))}^{\frac{G(x)}{1-G(x)}} \lambda\theta(\lambda t)^{\theta-1} e^{-(\lambda t)^{\theta}} dt, \delta \geq 0, x > 0.$$

- The Weibull class $\left(\delta G(x), \delta + \dfrac{1-G(x)}{G(x)}\right)$

$$H_G(x) = \int_{\delta G(x)}^{\frac{\delta+(1-G(x))}{G(x)}} \lambda\theta(\lambda t)^{\theta-1} e^{-(\lambda t)^{\theta}} dt, \delta \geq 0, x > 0.$$

2.7 Maximum likelihood

Let $(X_1, \ldots, X_n)^\top$ a random sample size of n random variable X with density function $f(x|\theta)$ with $\theta \in \Theta$, where Θ is the parameter space. The likelihood function of θ corresponding to the observed random sample is given by $L(\theta; x) = \prod_{i=1}^{n} f(x_i|\theta)$. The estimators of maximum likelihood are the values of θ that maximize the likelihood function $L(\theta; x)$ or their log-likelihood $\ell(\theta) = \log(L(\theta))$. The values are obtained by solving the system of equations supports vectors

$$U(\theta) = \frac{\partial L(\theta)}{\partial \theta}. \tag{2.5}$$

In this thesis, the Adequacy Model the statistical software R [55] was used to obtain the maximum likelihood estimates for all proposed distributions.

2.8 Criteria for choosing a model

Current literature presents many criteria for choosing models that are readily available on most computer programs, such as Akaike information criterion (AIC) [3], Bozdogan [9] the Corrected AIC (CAIC), the [62] Bayesian information criterion (BIC) and the Hannan and Quinn [30] Hannan-Quinn information criterion (HQIC). The main characteristic of these criteria is that their calculations are based on the logarithm of the value of the model's likelihood function and that they vary by a function of the number of observations and the number of model parameters.

2.8.1 AIC

Suppose we have many models of the probability distribution of a data set, and want to pick the one that is closest to the "real" data distribution. Let p be the number of estimated parameters on a given model, then define the quantity

$$AIC = -2\ell(\theta) + 2p$$

as the information loss between the true data distribution and the estimated distribution. Akaike [3] Akaike's analysis shows that, asymptotically, an automated process that picks models based on this quantity - today called Akaike's Information Criterion - minimizes the information loss, as measured by the Kullback-Leibler Divergence. Based on this analysis, the best model is the one that has the lowest AIC, and all the other selection criteria will follow the same logic.

2.8.2 CAIC

Bozdogan [9] proposed a correction of the AIC called CAIC. When there are many parameters compared to the sample size the AIC may exhibit poor performance, so, according to Burnham and Anderson [10], CAIC should be used when ($\frac{n}{p} < 40$). Where p is the number of parameters and n sample size. the best model is the one that has the lowest CAIC.

$$CAIC = AIC + \frac{2p(p+1)}{n-p-1}$$

2.8.3 BIC

The Bayesian information criterion (BIC), also known as Schwarz's Criterion [62], and differs from other models because it uses a posteriori probability. Let p be the number of parameters and n the sample size. The best model will be that one with the lowest value of BIC.

$$BIC = -2\ell(\theta) + p\log(n)$$

2.8.4 HQIC

The Hannan-Quinn Information Criterion [30] is a variant of the BIC for large sample sizes given by the expression

$$HQIC = 2p\log\left(\log(n)\right) - 2\log(L(\hat{\theta}))\cdot$$

Where p is the number of parameters, n sample size and $L(\hat{\theta})$ is the maximum likelihood function.

2.8.5 Tests of adherence

Other two statistics used for selecting models are Anderson-Darling (A^*) and Cramer-Von Mises (W^*) Chen and Balakrishnan [20].

$$A^* = \left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right) \times \left\{-n - \frac{1}{n}\sum_{i=1}^{n}\left[(2i-1)\log(p_i) + (2n+1-2i)\right]\frac{2n\log(1-p_i)+1}{2n}\right\},$$

where n is sample size and p_i are ordered percentiles and

$$W^* = \left(1 + \frac{0.5}{n}\right)\sum_{i=1}^{n}\left[u_i - \frac{(2i-1)}{2n}\right]^2 + \frac{1}{2n}.$$

Where $u_i = \Phi\left(\frac{[y_i - \bar{y}]}{s_y}\right)$, $y_i = \Phi^{-1}(\upsilon_i)$, $\upsilon_i = F\left(x_i; \hat{\theta}\right)$ and $\Phi(.)$ is of normal distribution.

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CHAPTER 3

New Classes of Trigonometric Distributions I: Sin-G Class

Resumo

Propomos uma nova classe de distribuições trigonométricas, denotada por **Sin-G** e uma nova distribuição como membro desta classe, a Seno Weibull Inversa que chamamos de **Sin-IW**, com apenas dois parâmetros. Analisamos as propriedades estatísticas da classe, incluindo expansões para a função de densidade, momentos e entropia de Rényi. Apresentamos um procedimento para obter as estimativas do modelo através do método de máxima verossimilhança. Adicionalmente, a matriz de informação observada é obtida. Realizamos uma aplicação do modelo em um conjunto de dados reais.

Palavras-chaves: Classes trigonométricas seno, Distribuição Weibull inversa; Estimação por máxima verossimilhança.

Abstract

A new class of trigonometric distributions is proposed, expressed by **Sin-G** and a new distribution is also presented, as part of this class, the inverse Weibull Sine class, called **Sin-IW**, which has only two parameters. The statistical properties of the class were analyzed, including expansion for the density function, moments and Rényi entropy. In addition, there is an introduction of a procedure which aims at collecting estimates of the model through the maximum likelihood method. Furthermore, the observed information matrix is obtained. Concluding, an application of the model to real life data set is carried out.

Keywords: Classes of trigonometric sine, Inverse Weibull distribution; Maximum likelihood estimation.

3.1 Introduction

Recently, Brito [3] developed techniques to construct a continuous of new classes of univariate and multivariate distributions and their properties. This work, as all of the work in the field of generalized distributions, can be seen today as a particular case of the results in Brito [3]. In a sense, all of the distributions are there for the development of statistical properties and for the use/applications of a particular generalized distribution or class. We use the following baselines for the construction of new classes of trigonometric probabilistic distributions:

$$H_G(x) = \int_0^{\frac{\pi}{2}\ G(x)} \cos(t)dt\cdot \tag{3.1}$$

The continuous distribution baseline $G(x)$ is a cdf. The cdf corresponding to (3.1) is

$$H_G(x) = \sin\left(\frac{\pi}{2}\ G(x)\right), \tag{3.2}$$

and if $G(x)$ has pdf $g(x)$ the pdf of the class given by

$$h_G(x) = \frac{\pi}{2} g(x) \cos\left(\frac{\pi}{2} G(x)\right)\cdot \tag{3.3}$$

We represent this class by **Sin-G**. The parameters of the new model will are the same of the probability distribution $G(x)$

3.1.1 The Hazard function the Sin-G class of distributions

If $G(x)$ is absolutely continuous, then the hrf of the **Sin-G** class of distributions is given by

$$R_G(x) = \frac{h_G(x)}{1 - H_G(x)}\cdot$$

Thus, the hrf of the proposed class is given by

$$R_G(x) = \frac{\frac{\pi}{2} g(x) \cos\left(\frac{\pi}{2} G(x)\right)}{1 - \sin\left(\frac{\pi}{2} G(x)\right)}\cdot \tag{3.4}$$

3.1.2 Quantile function

Some of the properties of distribution can be studied through its moments, quantiles, skewness and kurtosis. Also, quantiles can be utilized to obtain data of the distribution according.

$$x = Q(u) = F^{-1}(u) = G^{-1}\left[\frac{2}{\pi} \arcsin(u)\right]\cdot$$

Table 3.1: Quantile and random number generator

Algorithm Random generator for the **Sin-G** class
1. Generate $u \sim U(0,1)$.
2. Specify $G^{-1}(\cdot)$
3. Obtain an outcome of X by $X = Q(u)$

3.2 Useful expansion

Theorem 3.2.1. *If $G(x)$ is the cdf baseline of a random variable and $g(x)$ is its pdf, then the pdf of the class can be expressed in terms of the a linear combination of the pdf of exponentiated G distributions.*

$$h_G(x) = \sum_{k=0}^{\infty} s_k g_{(2k+1)}(x), \tag{3.5}$$

where $s_k = \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}$ *and* $g_{(2k+1)}(x)$ *denotes the exp-G pdf with parameters* $(2k+1)\cdot$

Proof. Writing the *sine* function in terms of his expansion in Taylor series, we will have

$$\sin\left[\frac{\pi}{2}G(x)\right] = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left[\frac{\pi}{2}G(x)\right]^{2k+1}.$$

For arbitrary baseline cdf $G(x)$, a random variable is said to have the exponentiated-G ($exp-G$) with power parameter $a > 0$. Which gives the pdf as,

$$h_G(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(2k+1)\left(\frac{\pi}{2}\right)^{2k+1} g(x)G^{(2k)}(x),$$

the above density function becomes

$$h_G(x) = \sum_{k=0}^{\infty} s_k\ g_{(2k+1)}(x),$$

where

$$s_k = \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}, \tag{3.6}$$

and $g_{(2k+1)}(x)$ denotes the pdf of the exponentiated G distribution with parameters $(2k+1)$. Clearly $\sum_{k=0}^{\infty} s_k = 1$. □

Similarly, we can obtain the expansion for the $H_G(x)$ by integration (3.5), since

Corollary 3.2.1. *If $G(x)$ is the cdf baseline of a random variable, then the cdf of the class can be expressed in terms of a linear combination of the cdf of the exponentiated G distributions.*

$$H_G(x) = \sum_{k=0}^{\infty} s_k\ G_{(2k+1)}(x),$$

where s_k is given by (3.6).

3.2.1 Moments of order m for the Sin-G class of distributions

In this section we will express the moment expansion of order m will be an infinite linear combination of moments of order m of exponentiated $G(x)$ distribution:

Theorem 3.2.2. *If $g(x)$ is pdf of a random variable, then the moments of order m of the* **Sin-G** *can be seen as a linear combination of moments of order m of exponentiated $G(x)$ distributions.*

$$\mu_m = \sum_{k=0}^{\infty} s_k \ \mu_m^{*k},$$

Proof. Let the moments of order m

$$\mu_m = E(X^m) = \int_0^{+\infty} x^m dH_G(x),$$

using the pdf expansion (3.5), we obtain

$$\mu_m = \int_0^{+\infty} x^m \sum_{k=0}^{\infty} s_k \ g_{(2k+1)}(x)dx, \tag{3.7}$$

which is equivalent to

$$\mu_m = \sum_{k=0}^{\infty} s_k \int_0^{+\infty} x^m g_{(2k+1)}(x)dx\cdot$$

The expansion of moments of order m can be seen as a linear combination of moments of order m of distributions $g_{(2k+1)}(x)$. Where s_k is given by (3.6).

$$\mu_m = \sum_{k=0}^{\infty} s_k \ \mu_m^{*k}\cdot$$

□

3.2.2 Central moments of order m for Sin-G class of distributions

Using the relationship between the central moment and non-central moments, we can calculate the expansion for central moments of order m for the proposed class as follows

Corollary 3.2.2. *The central moments of order m can be seen as a linear combination of moments of order m of distributions exponentiated $G(x)$ distributions.*

$$\mu'_m = \sum_{k=1}^{\infty} \gamma_k(m)\mu_m^{'(k)}, \tag{3.8}$$

where

$$\gamma_k(m) = \sum_{r=0}^{m} \binom{m}{r} \frac{(-1)^{r+k}\mu^r}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1}$$

and $\mu_m^{'(k)}$ is the m-th central moment of $Y_{(2k+1)}$. Considering $m = 2$, we get the expansion for variance of the proposed class.

$$\sigma^2 = \mu_2' = \sum_{k=1}^{\infty} \gamma_k(2)\mu_2^{'(k)}.$$

3.2.3 Generating and characteristic functions for Sin-G class

The moment generating function (mgf) is a very important measure as it is possible to obtain properties of the distribution as average, dispersion, symmetry and kurtosis. The mgf of the Class is given by the following theorem.

Theorem 3.2.3. *Consider $Y_{2k+1} \sim G_{2k+1}(x)$. Then the moment generating function (mgf) can be expressed as an infinite linear combination of the generating functions of moments of a $G(x)$ exponentiated distribution.*

$$M_X(t) = \sum_{k=0}^{\infty} s_k M_{Y_{(2k+1)}}(t).$$

Proof. Let the moment generating function

$$M_X(t) = E(e^{tX}) = \int_0^{+\infty} e^{tx} dH(x),$$

using the pdf expansion, (3.5) we obtain

$$M_X(t) = \int_0^{+\infty} e^{tx} \sum_{k=0}^{\infty} s_k \ g_{(2k+1)}(x)dx, \tag{3.9}$$

rewriting the mgf as infinite linear combination of the mgf of distributions $g_{(2k+1)}(x)$.

$$M_X(t) = \sum_{k=0}^{\infty} s_k \ M_{Y_{(2k+1)}}(t),$$

where $M_{Y_{(2k+1)}}(t)$ is the mgf of the exponentiated $g_{(2k+1)}(x)$ distributions and s_k is defined by Eq. (3.6). □

It can be seen that the characteristic function can be rewritten as infinite linear combinations of the characteristic functions of distributions $g_{(2k+1)}(x)$.

$$\varphi_X(t) = \sum_{k=0}^{\infty} s_k \ \varphi_{Y_{(2k+1)}}(t).$$

3.2.4 General coefficient for the Sin-G class of distributions

The general coefficient of a random variable X is given by

$$C_g(m) = \frac{\mu'_m}{\sigma^m}. \tag{3.10}$$

Thus, we have the following expression for the general coefficient of the **Sin-G** class of distributions:

$$C_g(m) = \frac{\sum_{k=1}^{\infty} \gamma_k(m)\mu_m^{'(k)}}{\left[\sum_{k=1}^{\infty} \gamma_k(2)\mu_2^{'(k)}\right]^{m/2}}. \tag{3.11}$$

So the asymmetry and kurtosis of the **Sin-G** can be respectively expressed by

$$C_g(3) = \frac{\sum_{k=1}^{\infty} \gamma_k(3)\mu_3^{'(k)}}{\left[\sum_{k=1}^{\infty} \gamma_k(2)\mu_2^{'(k)}\right]^{3/2}}, \tag{3.12}$$

and

$$C_g(4) = \frac{\sum_{k=1}^{\infty} \gamma_k(4)\mu_4^{'(k)}}{\left[\sum_{k=1}^{\infty} \gamma_k(2)\mu_2^{'(k)}\right]^{2}}. \tag{3.13}$$

3.3 Entropy for the Sin-G class of distributions

The entropy of a distribution is a measure of uncertainty; the greater the entropy, the higher the disorder and less likely to observe a given event; the lower the entropy, the lower its disorder and the higher the probability of observing a particular event. In this section, we obtain expansions for the Rényi [5] entropy. The entropy is given by

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma} \log\left(\int_{-\infty}^{+\infty} h_G^{\gamma}(x)dx\right). \tag{3.14}$$

By expanding $W(s) = \cos^{\gamma}\left[\frac{\pi}{2}\, s\right]$ in Taylor series, we have

$$\cos^{\gamma}\left[\frac{\pi}{2}\, s\right] = \sum_{k=0}^{\infty}\sum_{r=0}^{k}(-1)^{k}\, a_k \binom{k}{r}\, s^{r}, \tag{3.15}$$

where $a_k = \frac{(-1)^k\, W^{(k)}(1)}{k!}$, and $W^{(k)}(1)$ denotes the kth derivative of $W(\cdot)$ evaluated at the point 1. Next, we derive expressions for the Rényi entropy for the Sin-G class. Due to the fact that the parameter γ is not in general a natural number, it is difficult to use Equation (3.5). Substituting (3.3) and (3.15) into Equation (3.14), and after some algebra, we obtain

$$\mathfrak{L}_{R,G}(\gamma) = \frac{\gamma}{1-\gamma}\left\{\log(\pi/2)^{\gamma} + \log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k\,(-1)^{r}\binom{k}{r}\,\mathrm{I}_r\right]\right\}, \tag{3.16}$$

where I_r comes from baseline distribution as

$$\mathrm{I}_r = \int_{-\infty}^{+\infty} G^{r}(x)\, g^{\gamma}(x)\, \mathrm{dx}\cdot$$

The quantity I_r can be computed at least numerically. The Shannon entropy is given by $E\{-\log[f(X)]\}$. It is a special case of the Rényi entropy when $\gamma \uparrow 1$.

3.4 Getting the score function for the Sin-G class of distributions

Suppose $\underset{\sim}{x} = x_1, \ldots, x_n$ is a random sample from **Sin-G** with parameter θ. Then the log-likelihood (LL) function for class is as it can be seen below:

$$\begin{aligned}\ell(\underset{\sim}{\theta}) = n\log\frac{\pi}{2} + \sum_{i=1}^{n}\log\left(g(x_i|\underset{\sim}{\theta})\right)\\ + \sum_{i=1}^{n}\log\left(\cos\left(\frac{\pi}{2}G(x_i|\underset{\sim}{\theta})\right)\right)\cdot\end{aligned}$$

obtaining a score function

$$U(\theta_j) = \frac{\partial\ell(\underset{\sim}{\theta})}{\partial\theta_j},$$

and a score function for class in θ_j, for $j = 1, \ldots, p$ and $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_p)$ and p is the number of parameters of the distribution $G(x|\underset{\sim}{\theta})$.

$$U(\theta_j) = \sum_{i=1}^{n} \frac{1}{g(x_i|\underset{\sim}{\theta})} \frac{\partial g(x_i|\underset{\sim}{\theta})}{\partial \theta_j} + \frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial \theta_j} \quad - \sum_{i=1}^{n} \left(\frac{\pi}{2} G(x_i|\underset{\sim}{\theta})\right) \tan\left(\frac{\pi}{2} G(x_i|\underset{\sim}{\theta})\right) \frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial \theta_j}.$$

3.5 The Sin-IW distributions

Considering that $G(x)$ is the IW distribution (3.2), the fdp corresponding to Equation (3.3), for $x > 0$, is given by

$$h_{IW}(x) \quad = \quad \frac{\alpha\theta\pi}{2} x^{-\theta-1} \exp(-\alpha x^{-\theta}) \cos\left[\frac{\pi}{2} \exp(-\alpha x^{-\theta})\right]. \tag{3.17}$$

The cdf corresponding to Eq. (3.2) for $x > 0$, is given by

$$H_{IW}(x) \quad = \quad \sin\left[\frac{\pi}{2} \ \exp(-\alpha x^{-\theta})\right] \cdot$$

We represent this distribution by **Sin-IW**. Finally, the hrf corresponding to (3.4) is given by

$$R_{IW}(x) = \frac{\frac{\alpha\theta\pi}{2} x^{-\theta-1} \exp(-\alpha x^{-\theta}) \cos\left[\frac{\pi}{2} \exp(-\alpha x^{-\theta})\right]}{1 - \sin\left[\frac{\pi}{2} \ \exp(-\alpha x^{-\theta})\right]}. \tag{3.18}$$

Plots of the pdf (3.1a), cdf (3.1b), the hrf (3.1c) and the survival function (3.1d) for selected parameter values are shown in Figure 3.1. The importance of the hazard rate function (3.1c) is quite flexible for modeling survival data. For selected parameter values, the **Sin-IW** distribution becomes an unimodal hazard rate function and decreasing hazard rate function.

3.6 Useful expansion

In the following sections we use the expansions defined for the **Sin-G** class of distributions in the development of the distribution **Sin-IW**. Considering that $G(x)$ is the IW distribution, that the cdf of **Sin-IW** is given by equation (3.19), the **Sin-IW** cdf expansion

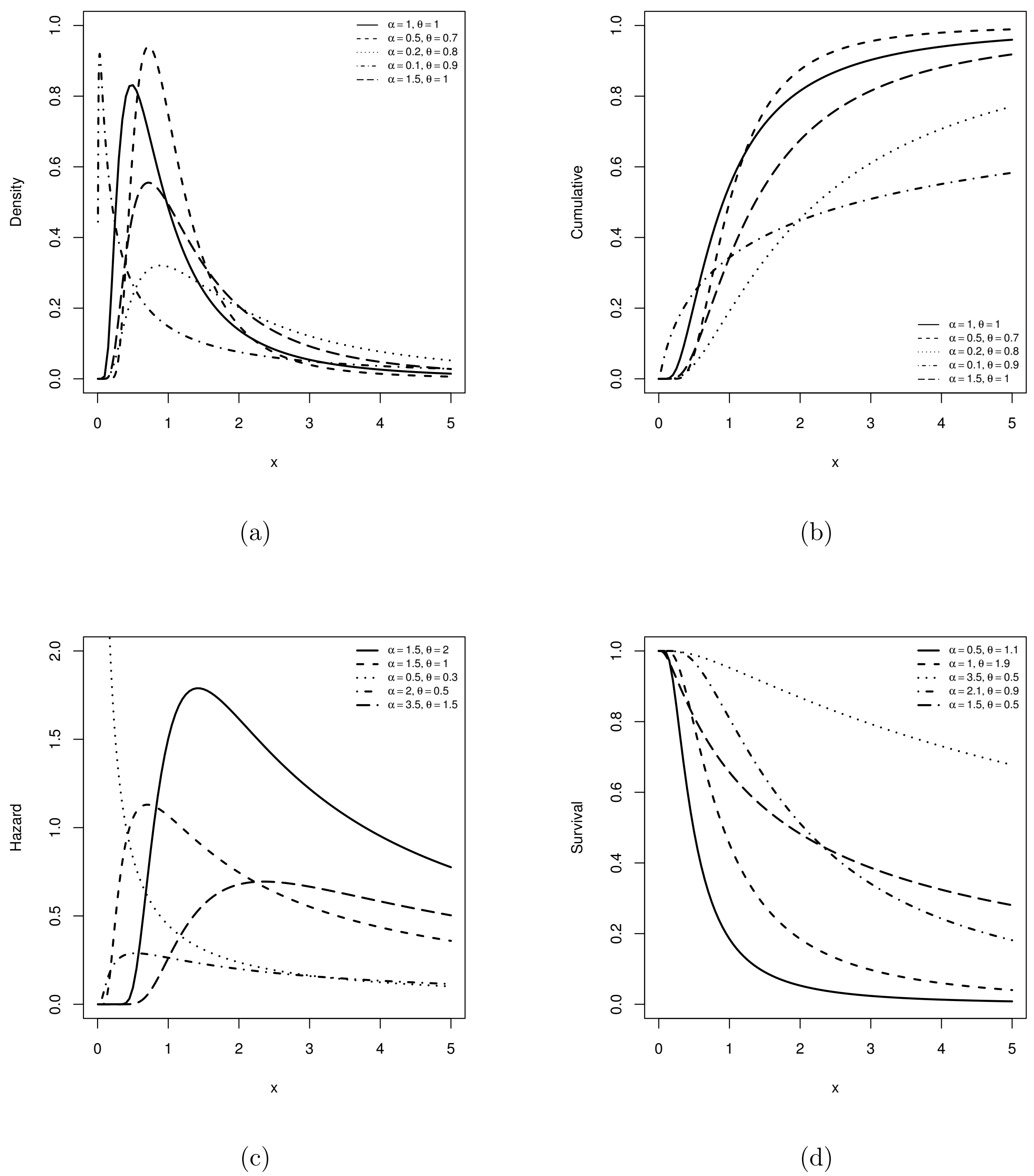


Figure 3.1: Plots of the (a) Sin-IW density, (b) Sin-IW cumulative, (c) Sin-IW hazard rate function and (d) Sin-IW survival

is given by

$$H_{IW}(x) = \sum_{k=0}^{\infty} s_k \; G_{(2k+1)}(x). \tag{3.19}$$

Where $s_k = \dfrac{(-1)^k}{(2k+1)!}\left(\dfrac{\pi}{2}\right)^{2k+1}$ and $G_{(2k+1)}(x) = \exp\left[-(2k+1)\alpha x^{-\theta}\right]$ are cdf distributions. Similarly, we can obtain the expansion for $h_{IW}(x)$, as follows

$$
\begin{aligned}
h_{IW}(x) &= \sum_{k=0}^{\infty} s_k (2k+1)\alpha\theta x^{-\theta-1} \exp\left[-(2k+1)\alpha x^{-\theta}\right] \\
h_{IW}(x) &= \sum_{k=0}^{\infty} s_k \; g_{(2k+1)}(x). \qquad (3.20)
\end{aligned}
$$

Where $g_{(2k+1)}(x)$ is the IW density function with parameters $(2k+1)$, α and θ. The pdf (3.20) can be expressed as a linear combination of IW.

3.6.1 Moments of order m for the Sin-IW distribution

Using an Eq. (3.7), we introduce the moment expansion of order m for the **Sin-IW** distribution.

$$\mu_m = \sum_{k=0}^{\infty} \omega_k \int_0^{+\infty} (2k+1)\alpha\theta x^{(m-\theta-1)} \exp\left[-(2k+1)\alpha x^{-\theta}\right] dx,$$

on substituting $y = (2k+1)\alpha x^{-\theta}$, the integral on the right reduces to

$$\mu_m = \sum_{k=0}^{\infty} \omega_k (2k+1)(2k+1)^{\frac{m}{\theta}-1} \alpha^{\frac{m}{\theta}} \int_0^{+\infty} y^{-\frac{m}{\theta}} \exp(-y) dy,$$

thus, it is immediate that the following equation is achieved

$$\mu_m = \sum_{k=0}^{\infty} \omega_k (2k+1)(2k+1)^{\frac{m}{\theta}-1} \alpha^{\frac{m}{\theta}} \Gamma\left(1 - \frac{m}{\theta}\right)$$

3.6.2 Generating moment and characteristic functions for Sin-IW

Using an identical manipulation to the one used in (3.9) the moment function is given by

$$M_X(t) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_k \left[\frac{t\,(2\alpha k)^{\frac{1}{\theta}}}{m!}\right]^m \Gamma\left(1 - \frac{m}{\theta}\right),$$

Using an Eq. (3.11) the characteristic function is given by

$$\varphi_X(t) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_k \left[\frac{it\,(2\alpha k)^{\frac{1}{\theta}}}{m!}\right]^m \Gamma\left(1 - \frac{m}{\theta}\right).$$

3.6.3 Quantile function for Sin-IW

The **Sin-IW** quantile function, can be obtained by inverting (3.17). Where

$$x = Q(u) = F^{-1}(u) = \left\{ -\frac{\log\left[\frac{2}{\pi}\arcsin(1-u)\right]}{\alpha} \right\}^{-\frac{1}{\theta}}.$$

3.6.4 Expansion to the central moments of order m for Sin-IW distribution

Using Eq. (3.8) we achieve the following

$$\mu'_m = \sum_{k=0}^{\infty}\sum_{r=0}^{m} \binom{m}{r}\left(\frac{\pi}{2}\right)^{2k+1} \frac{\mu^r(-1)^{k+r}\alpha\theta}{(2k+1)!}(2k+1)^{\frac{m-r}{\theta}-1}\alpha^{\frac{m-r}{\theta}}\Gamma\left(1-\frac{m-r}{\theta}\right)$$

$$\mu'_m = \sum_{k=0}^{\infty}\sum_{r=0}^{m} b_{k,r}(m)\Gamma\left(1-\frac{m-r}{\theta}\right).$$

Where $b_{k,r}(m) = \binom{m}{r}\left(\frac{\pi}{2}\right)^{2k+3}\frac{\mu^r(-1)^{k+r+1}\alpha\theta}{(2k+1)!}(2k+1)^{\frac{m-r}{\theta}-1}\alpha^{\frac{m-r}{\theta}}$. Considering $m = 2$, we get the expansion for variance of the distribution.

$$\sigma^2 = \mu'_2 = \sum_{k=0}^{\infty}\sum_{r=0}^{2} b_{k,r}(2)\Gamma\left(1-\frac{2-r}{\theta}\right).$$

3.6.5 Expansion to the general rate for the Sin-IW distribution

Using Eq. (3.10), the result achieved is:

$$C_g(m) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{m} b_{k,r}(m)\Gamma\left(1-\frac{m-r}{\theta}\right)}{\left[\sum_{k=0}^{\infty}\sum_{r=0}^{2} b_{k,r}(2)\Gamma\left(1-\frac{2-r}{\theta}\right)\right]^{m/2}}.$$

The asymmetry and kurtosis (3.12) and (3.13) can be respectively expressed by

$$C_g(3) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{3} b_{k,r}(3)\Gamma\left(1-\frac{3-r}{\theta}\right)}{\left[\sum_{k=0}^{\infty}\sum_{r=0}^{2} b_{k,r}(2)\Gamma\left(1-\frac{2-r}{\theta}\right)\right]^{3/2}}.$$

and

$$C_g(4) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{4} b_{k,r}(4)\Gamma\left(1-\frac{4-r}{\theta}\right)}{\left[\sum_{k=0}^{\infty}\sum_{r=0}^{2} b_{k,r}(2)\Gamma\left(1-\frac{2-r}{\theta}\right)\right]^2}.$$

3.7 Entropy for the Sin-IW distribution

The entropy of a random variable is a measure of uncertain variation [6]. Based on the manipulation used in (3.18) and pdf (3.5), the Rényi [5] entropy is given by

$$\mathfrak{L}_{R,G}(\delta) = \frac{1}{1-\delta}\log\left[\int_0^{+\infty}\left(\frac{\alpha\theta\pi}{2}\right)^{\delta} x^{-(\theta+1)\delta}\exp(-\delta\alpha x^{-\delta})\cos^{\delta}\left[\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right]\right]dx\cdot$$

By using Taylor series we can write $W(s) = \cos^{\delta}\left(\frac{\pi}{2}s\right) = \sum_{k=0}^{\infty} a_k(1-s)^k = \sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k\binom{k}{r}(-1)^r s^r$ and hence

$$\mathfrak{L}_{R,G}(\delta) = \frac{1}{1-\delta}\log\left[\left(\frac{\alpha\theta\pi}{2}\right)^{\delta}\sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k(-1)^r\binom{k}{r}\int_0^{+\infty} x^{-(\theta+1)\delta}\exp(-(\delta+k)\alpha x^{-\theta})\right]dx \quad (3.21)$$

where $a_k = \dfrac{(-1)^k W^{(k)}(1)}{k!}$.

Setting $y = (\delta+k)\alpha x^{-\theta}$ in Eq. (3.21) for solve last integral, we have

$$\mathfrak{L}_{R,G}(\delta) = \frac{1}{1-\delta} \times \log\left\{(\alpha\theta)^{\delta-1}\left(\frac{\pi}{2}\right)^{\delta}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r a_k}{(\delta+k)^{\frac{(\theta+1)(\delta-1)}{\theta}+1}}\Gamma\left[\frac{(\theta+1)(\delta-1)}{\theta}+1\right]\right\}.$$

3.8 Maximum likelihood estimation

We obtain estimation by the method of maximum likelihood estimator (MLE) of the parameters α and θ of (3.5). Let $\mathbf{x} = \{x_1,\ldots,x_n\}^T$ of size n of independent random variables from the $\mathbf{Sin-IW}$ distribution with parameter vector $\omega = (\alpha,\theta)^T$. The log-likelihood for ω

is given by

$$L = n\log\left(\frac{\pi}{2}\right) + n\log(\alpha\theta) - (\theta+1)\sum_{i=1}^{n}\log(x_i)$$
$$+ n\log\left[\cos\left(\frac{\pi}{2}\right)\exp(-\alpha x_i^{-\theta})\right]\cdot \tag{3.22}$$

The elements of the score vector $U(\omega)$ are given by

$$U_\alpha = \frac{n}{\alpha} + \frac{\pi}{2}\sum_{i=1}^{n} x_i^{-\theta}\tan\left[\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right]\exp(-\alpha x_i^{-\theta})$$

and

$$U_\theta = \frac{n}{\theta} - \sum_{i=1}^{n}\log(x_i) - \frac{\alpha\pi}{2}\sum_{i=1}^{n} x_i^{-\theta}\log(x_i)\tan\left[\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right]\exp(-\alpha x_i^{-\theta})\cdot$$

3.9 Application

Here we use the **Sin-IW** distribution in an application to a real data set. We shall compare to Weibull exponential Eq. (3.23), Beta exponential Eq. (3.24) and Weibull Eq. (3.25) respectively. Its density functions are given by

$$f(x;k,\lambda;\alpha) = \alpha\frac{k}{\lambda}\left[\frac{x}{\lambda}\right]^{k-1}\left[1-\exp\left(-\frac{x}{\lambda}\right)^{k}\right]\exp\left(-\frac{x}{\lambda}\right)^{k}, \tag{3.23}$$

$$f(x) = \frac{\lambda}{B(a,b)}\exp(-b\lambda x)\left[1-\exp(-\lambda x)\right]^{a-1} \tag{3.24}$$

and

$$f(x) = \alpha\lambda^{\alpha}x^{\alpha-1}\exp\left(-(\lambda x)^{\alpha}\right), \tag{3.25}$$

where $a > 0, \alpha > 0$, $b > 0, k > 0$, $\lambda > 0$ and $x > 0$. The data set from Bjerkedal [2] of 72 observations represents the survival times, in days of guinea pigs injected with different doses of tubercle bacilli. Are listed below in Table 3.2. Table 3.3 displays some descriptive statistics. Table 3.4 lists the MLEs of the parameters (standard errors in parentheses). We

Table 3.2: Guinea Pigs Data

12	15	22	24	24	32	32	33	34	38	38	43	44	48
52	53	54	54	55	56	57	58	58	59	60	60	60	60
61	62	63	65	65	67	68	70	70	72	73	75	76	76
81	83	84	85	87	91	95	96	98	99	109	110	121	127
129	131	143	146	146	175	175	211	233	258	258	263	297	341
341	376												

Table 3.3: Descriptive statistics.

Min.	Q_1	Median	Mean	Q_3	Max.	Var.
12.00	54.75	70.00	99.82	112.80	376.00	6580.122

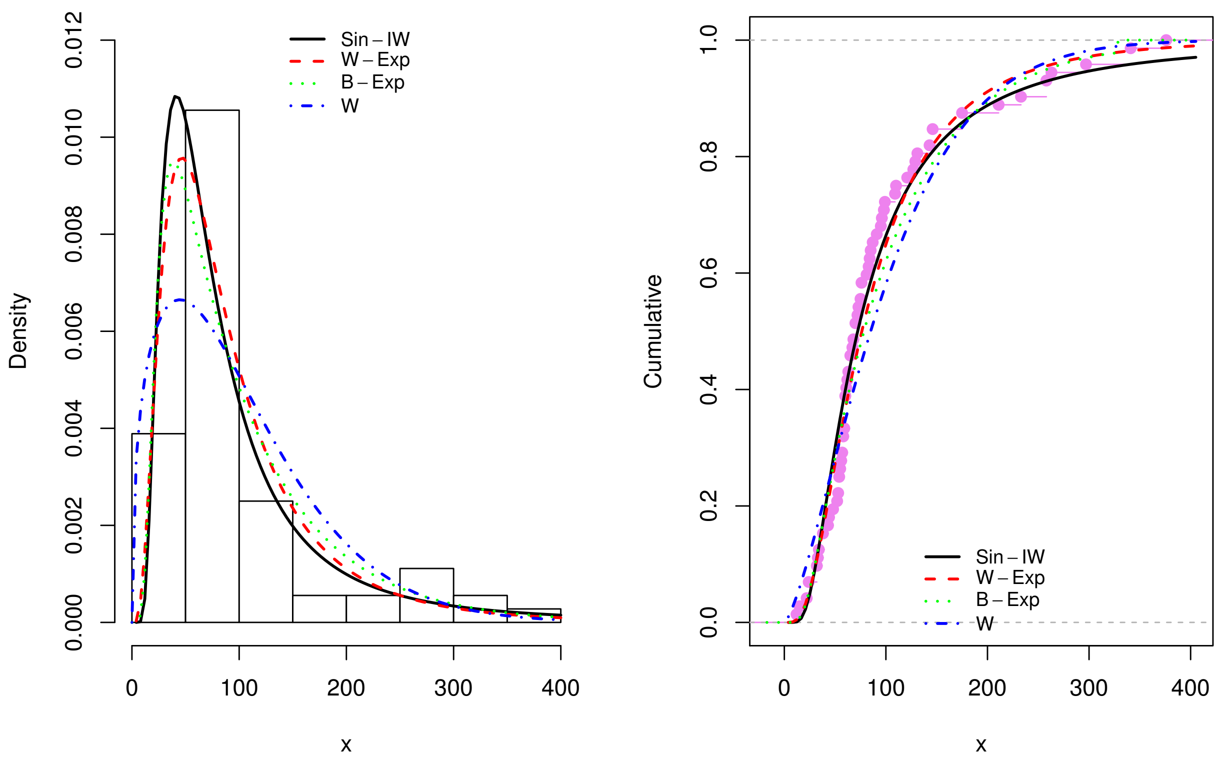
see that the new distribution, when compared to others, as well as the following statistics: AIC, BIC Akaike [1], and statistics Anderson-Darling [4] provided better statistics, according to A^* and W^*. Thus, we conclude that this distribution is quite flexible in the modeling of the proposed data. So, we can say that the Fig. 3.2 suggests an excellent fit to the data distribution to the adequacy of the data.

Table 3.4: Estimates of Models for Guinea Pigs Data

Distributions	Estimates			AIC	BIC	CAIC	HQIC	A^*	W^*
Sin-IW (α, θ)	115.12	1.09	–	787.66	792.21	787.83	789.47	0.81	0.14
	(41.96)	(0.09)	–						
W-Exp (α, k, λ)	15.36	0.51	8.27	786.31	793.14	786.67	789.03	0.81	0.15
	(10.21)	(0.10)	(7.11)						
B-Exp (a, b, λ)	10.09	0.11	0.11	789.09	795.93	789.45	791.81	7.36	1.26
	(9.22)	(0.07)	(0.06)						
W (α, λ)	1.39	0.01	–	798.29	802.84	798.47	800.10	2.39	0.43
	(0.12)	(0.00)	–						

3.10 Concluding remarks

We proposed a new class of trigonometric distribution, the **Sin-G**, and a new distribution in this class, the Sine Inverse Weibull distribution. We obtain the density function, cumulative function and its expansions. In addition, the entropy was calculated and their estimates were checked via the maximum likelihood method. Plots of the estimated pdf and cdf indicate that **Sin-IW** distribution is superior to the other distributions. In Figure (3.2) we can see that this model can help in the analysis of survival data, as well as in other areas of knowledge.



(a) (b)

Figure 3.2: Some estimated fitted (a) densities and (b) cumulative of the data.

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CHAPTER 4

New Classes of Trigonometric Distributions II: **Cos-G**

Resumo

Apresentamos uma nova classe de distribuições trigonométricas chamada **Cos-G** e uma nova distribuição, a Cosseno Weibull nominada **Cos-W** , com apenas dois parâmetros. Damos um tratamento matemático da classe, incluindo expansões para a função de densidade, momentos, e entropia. Apresentamos um procedimento para estimar os parâmetros através método de máxima verossimilhança. Além disso, a informação observada é obtida. Em seguida, aplicamos o modelo a um conjunto de dados reais.

Palavras-chaves: Classes trigonométricas cosseno, Distribuição Weibull; Estimação por máxima verossimilhança.

Abstract

A new class of trigonometric distributions, called **Cos-G**, and a new distribution, the Weibull Cosine, **Cos-W**, having only two parameters, are presented. A Mathematical approach is applied to the class, including expansions for the density function, moments and entropy. In addition, a procedure for measuring parameters, through the maximum likelihood method, is presented. Furthermore, the observed information matrix is acquired. Then, the model is applied to a real life data set.

Keywords: Classes of trigonometric cosine, Inverse Weibull distribution; Maximum likelihood estimation.

4.1 Introduction

Recently, Brito [2] developed techniques to construct new classes of univariate and multivariate distributions. We use that as a baseline proposal for the construction of new classes of trigonometric probabilistic distributions. Let $G(x)$ be a cdf and the **Cos-G** class to have a cdf given by

$$H_G(x) \quad = \quad 1 - \int_{\frac{\pi}{2}\ G(x)}^{\frac{\pi}{2}} \sin(t)dt,$$

which can be rewritten as

$$H_G(x) \quad = \quad 1 - \cos\left(\frac{\pi}{2}\ G(x)\right)\cdot$$

If $G(x)$ is a cdf of continuous random variable, and its pdf $g(x)$, the corresponding pdf of the class is given by

$$h_G(x) \quad = \quad \frac{\pi}{2} g(x) \sin\left(\frac{\pi}{2} G(x)\right)\cdot \tag{4.1}$$

If $G(x)$ is absolutely continuous cdf, then its hrf is given by

$$R_G(x) = \frac{h_G(x)}{1 - H_G(x)}\cdot$$

Thus, we can obtain the hrf of the **Cos-G** class of distributions as presented below

$$R_G(x) = \frac{\pi}{2} g(x) \tan\left(\frac{\pi}{2} G(x)\right)\cdot$$

4.1.1 Quantile function

Some of the properties of distribution can be studied through its moments, quantiles, skewness and kurtosis. Also, quantiles can be utilized to obtain data of the distribution according.

$$x = Q(u) = F^{-1}(u) = G^{-1}\left[\frac{2}{\pi}\arccos(1-u)\right]$$

Table 4.1: Quantile and random number generator

Algorithm Random generator for the **Cos-G** class
1. Generate $u \sim U(0,1)$.
2. Specify $G^{-1}(\cdot)$
3. Obtain an outcome of X by $X = Q(u)$

4.2 Useful expansion

Now we introduce an expansion series of the accumulated function and the density function. This expansion allows the description of a variety of structural properties such as the moments, central moments and the moment generating function for the class.

Theorem 4.2.1. *If $G(x)$ is the baseline cdf of a random variable, then the cdf of the class can be expressed in terms of a linear combination of the cdf of exponentiated G distributions.*

$$H_G(x) = 1 - \sum_{k=0}^{\infty} b_{k+1} G_{2k+2}(x), \tag{4.2}$$

where $b_{k+1} = \frac{(-1)^{k+1}}{(2k+2)!}\left(\frac{\pi}{2}\right)^{2k+2}$ *and $G_{2k+2}(x)$ is exp-G.*

Proof. Using the Taylor series of the cosine function, we get

$$\cos\left[\frac{\pi}{2}G(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left[\frac{\pi}{2}G(x)\right]^{2k}.$$

The distribution $G(x)$ is exponentiated to the $2k$-th power, we set to $G_{2k}(x) = G^{2k}(x)$. The corresponding pdf is expressed by $g_{2k}(x) = 2kg(x)G^{2k-1}(x)$. Thus, can obtain the expansion for $H_G(x)$ as seen below

$$H_G(x) = 1 - \sum_{k=0}^{\infty} b_k G_{2k}(x),$$

hence, note that for $k = 0$ we have the first term equal to zero and then rewritten $H_G(x)$ as depicted below

$$H_G(x) = 1 - \sum_{k=0}^{\infty} b_{k+1} G_{2k+2}(x),$$

where $b_{k+1} = \dfrac{(-1)^{k+1}}{(2k+2)!}$ and $G_{2(k+1)}(x) \left(\dfrac{\pi}{2}\right)^{2k+2}$ denotes a cdf of the exponentiated G distribution with parameters $2(k+1)$. Clearly, $\sum_{k=0}^{\infty} b_{k+1} = 1$. □

Theorem 4.2.2. *If $G(x)$ is the cdf baseline of a random variable and $g(x)$ is its pdf, then the pdf of the class can be expressed in terms of a linear combination of the pdf of exponentiated G distributions.*

$$h_G(x) = \sum_{k=0}^{\infty} b_{k+1} g_{2(k+1)}(x), \tag{4.3}$$

Proof. We can obtain the expansion using Theorem 4.2.2

$$h_G(x) = \sum_{k=0}^{\infty} b_{k+1} 2(k+1) g(x) G^{2(k+1)-1}(x),$$

$$h_G(x) = \sum_{k=0}^{\infty} b_{k+1} g_{2(k+1)}(x) \cdot \tag{4.4}$$

where $g_{2(k+1)}(x)$ denotes a pdf of the exponentiated G distribution with parameters $2(k+1)$. □

4.3 General properties

In this section, we study some properties of the class. Writing the equations (4.2) and (4.3) in terms of a linear combination of the exponentiated pdf allows to obtain many results for this class. Some mathematical properties such as ordinary moments, central moments, the moment generating function, characteristic function and entropy class can be expressed as a linear combination of these measures.

4.3.1 Moments of order m

The moment's expansion of order m for the class can be obtained as depicted below

$$\mu_m = E(X^m) = \int_0^{+\infty} x^m dH_G(x),$$

using the expansion for the pdf,

Theorem 4.3.1. *If $Y_{2k+2} \sim G_{2k+2}(x)$ and, according to Theorem 4.2.2, we can rewrite the moment of order m as an infinite linear combination of moments of order m of the exponentiated G distributions.*

$$\mu_m = \sum_{k=0}^{\infty} b_{k+1} E\left[Y_{2(k+1)}^m\right],$$

Proof. the equation above is equivalent to

$$\begin{aligned}
\mu_m &= \int_0^{+\infty} \sum_{k=0}^{\infty} b_{k+1} x^m g_{2(k+1)}(x) dx, \\
\text{where} \\
\mu_m &= \sum_{k=0}^{\infty} b_{k+1} \int_0^{+\infty} x^m g_{2(k+1)}(x) dx, \\
\text{thus, we can rewrite} \\
\mu_m &= \sum_{k=0}^{\infty} b_{k+1} E\left[Y_{2(k+1)}^m\right] \qquad (4.5)
\end{aligned}$$

where $Y_{2(k+1)}$ denotes the random variable with pdf $g_{2(k+1)}(x)$. We can write the moments as an infinite linear combination of moments of exponentiated $G(x)$ distributions. □

4.3.2 Moment generating and characteristic functions

We derive the expansion for the moment generating function (mgf) of the **Cos-G** distribution moments. We get the following equation.

Theorem 4.3.2. *If $Y_{2k+2} \sim G_{2k+2}(x)$, then we can rewrite the mgf in terms of an infinite linear combination of mgf of exponentiated G distributions .*

$$M_X(t) = \sum_{k=0}^{\infty} \nu_k M_{Y_{2(k+1)}}(t),$$

Proof. By using the Taylor series expansion of the exponential in Eq. (4.6), we will have

$$e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!}, \tag{4.6}$$

$$M_X(t) = E(e^{tX}) = \int_0^{+\infty} e^{tx} dH_G(x),$$

by using the expansion for pdf of the **Cos-G** class of distributions we obtain

$$\begin{aligned} M_X(t) &= \sum_{k=0}^{\infty} b_{k+1} \int_0^{+\infty} e^{tx} g_{2(k+1)}(x)dx, \\ M_X(t) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} b_{k+1} \int_0^{+\infty} \frac{x^m t^m}{m!} g_{2(k+1)}(x)dx, \\ M_X(t) &= \sum_{k=0}^{\infty} \nu_k M_{Y_{2(k+1)}}(t), \end{aligned} \tag{4.7}$$

where $\nu_k = \sum_{m=0}^{\infty} b_{k+1}$ and $M_{2(k+1)}(t)$ is mgf of $Y_{2(k+1)}$ denotes the random variable with pdf $g_{2(k+1)}(x)$. We can write the moment generating function as an infinite linear combination of moment generating function of the exponentiated G distributions. □

Using an approach similar to the one used to obtain the expansion of the moment generating function, the following is achieved:

$$\varphi_X(t) = \sum_{k=0}^{\infty} b_{k+1} \varphi_{Y_{2(k+1)}}(t),$$

where $\varphi_{Y_{2(k+1)}}(t)$ is characteristic function of the pdf $g_{2(k+1)}(x)$. For the ordinary moments of order m we have

$$\mu'_m = \sum_{k=1}^{\infty} \gamma_k(m) \mu_m^{\prime(k)},$$

where $\gamma_k(m) = \sum_{r=0}^{m} \binom{m}{r} \frac{(-1)^{r+k}\mu^r}{(2k-1)!}\left(\frac{\pi}{2}\right)^{2k}$ and $\mu_m^{\prime(k)}$ is the m-th central moment of $Y_{2(k+1)}$. Considering $m = 2$, we get the expansion for variance of the proposed class.

$$\sigma^2 = \mu'_2 = \sum_{k=1}^{\infty} \gamma_k(2) \mu_2^{\prime(k)}.$$

4.3.3 General coefficient

The general coefficient of a random variable X is given by

$$C_g(m) = \frac{\mu'_m}{\sigma^m}.$$

Thus, we can write the general coefficient of the **Cos-G** class as

$$C_g(m) = \frac{\sum_{k=1}^{\infty} \gamma_k(m)\mu_m^{'(k)}}{\left[\sum_{k=1}^{\infty} \gamma_k(2)\mu_2^{'(k)}\right]^{m/2}}.$$

So, the asymmetry can be expressed as

$$C_g(3) = \frac{\sum_{k=1}^{\infty} \gamma_k(3)\mu_3^{'(k)}}{\left[\sum_{k=1}^{\infty} \gamma_k(2)\mu_2^{'(k)}\right]^{3/2}},$$

and the kurtosis is given by

$$C_g(4) = \frac{\sum_{k=1}^{\infty} \gamma_k(4)\mu_4^{'(k)}}{\left[\sum_{k=1}^{\infty} \gamma_k(2)\mu_2^{'(k)}\right]^{2}}.$$

4.3.4 Entropies

Entropy measures the uncertainty, Shannon [7], the greater the entropy, the higher the disorder and the less likely to observe a meaningful phenomenon; also the lower the entropy, the lower its disorder and the higher its probability of observing a particular event. Thus, we obtain expansions for the Rényi [6] entropy. Rényi is given by

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma} \log \left[\int_{-\infty}^{+\infty} h_G^{\gamma}(x) d(x)\right].$$

By expanding $W(s) = \sin^{\gamma}\left[\frac{\pi}{2}\, s\right]$ in Taylor series, we have

$$\sin^{\gamma}\left[\frac{\pi}{2}\, s\right] = \sum_{k=0}^{\infty}\sum_{r=0}^{k}(-1)^k\, a_k \binom{k}{r}\, s^r, \tag{4.8}$$

where $a_k = \frac{(-1)^k\, W^{(k)}(1)}{k!}$, and $W^{(k)}(1)$ denotes the kth derivative of $W(\cdot)$ evaluated at the point 1. Next, we derive expressions for the Rényi entropy for the Cos-G class. Due to the fact that the parameter γ is not in general a natural number, it is difficult to use Equation (4.3). Substituting (4.1) and (4.9) into Equation (4.8), and after some algebra, we obtain

$$\mathfrak{L}_{R,G}(\gamma) = \frac{\gamma}{1-\gamma}\left\{\log(\pi/2)^{\gamma} + \log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k\,(-1)^r \binom{k}{r}\, \mathrm{I}_r\right]\right\}, \tag{4.9}$$

where I_r comes from baseline distribution as

$$\mathrm{I}_r = \int_{-\infty}^{+\infty} G^r(x)\, g^{\gamma}(x)\, \mathrm{dx}.$$

The quantity I_r can be computed at least numerically. The Shannon entropy is given by $E\{-\log[f(X)]\}$. It is a special case of the Rényi entropy when $\gamma \uparrow 1$.

4.4 Getting the Score Function for Cos-G class of distributions

Let $\underset{\sim}{x} = (x_1, \cdots, x_n)^{\top}$ be a random sample of size n from the **Cos-G** with unknown parameter vector θ. Then the log-likelihood (LL) function for θ is as it follows:

$$\begin{aligned}\ell(\underset{\sim}{\theta}) = n\log\frac{\pi}{2} + \sum_{i=1}^{n}\log\Big(g(x_i|\underset{\sim}{\theta})\Big) \\ + \sum_{i=1}^{n}\log\left[\sin\left(\frac{\pi}{2}G(x_i|\underset{\sim}{\theta})\right)\right]\cdot\end{aligned}$$

obtaining a score function

$$U(\theta_j) \;=\; \frac{\partial\ell(\underset{\sim}{\theta})}{\partial\theta_j},$$

and a score function for class in θ_j, for $j = 1, \cdots, p$ and $\underset{\sim}{\theta} = (\theta_1, \cdots, \theta_p)$ and p is the number of parameters of the distribution $G(x|\underset{\sim}{\theta})$.

$$
\begin{aligned}
U(\theta_j) = & \sum_{i=1}^{n} \frac{1}{g(x_i|\underset{\sim}{\theta})} \frac{\partial g(x_i|\underset{\sim}{\theta})}{\partial \theta_j} + \sum_{i=1}^{n} \frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial \theta_j} \\
& + \sum_{i=1}^{n} \cot\left(\frac{\pi}{2} G(x_i|\underset{\sim}{\theta})\right) \frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial \theta_j} \cdot
\end{aligned}
$$

4.5 Distribution Cos-W proposal

If, in 4.1, we take $G(x)$ as the cdf of a W distribution, then we have the following pdf

$$
h_W(x) = \frac{\pi}{2} \alpha \lambda^{\alpha} x^{\alpha-1} \exp\left[-(\lambda x)^{\alpha}\right] \cos\left(\frac{\pi}{2} \exp\left[-(\lambda x)^{\alpha}\right]\right), x > 0 \cdot \quad (4.10)
$$

The corresponding cdf is given by

$$
H_W(x) = 1 - \sin\left(\frac{\pi}{2} \exp\left[-(\lambda x)^{\alpha}\right]\right), x > 0 \cdot
$$

From now on, we represent the distribution by **Cos-W**. Thus, hrf is given by

$$
R_W(x) = \frac{\pi}{2} \alpha \lambda^{\alpha} x^{\alpha-1} \exp\left[-(\lambda x)^{\alpha}\right] \cot\left(\frac{\pi}{2} \exp\left[-(\lambda x)^{\alpha}\right]\right) \cdot
$$

Plots of the pdf, cdf, hrf and survival, respectively for selected parameter values, are displayed in Figure 4.1. We can observe in the Hazard curves shapes decreasing, creasing and also unimodal.

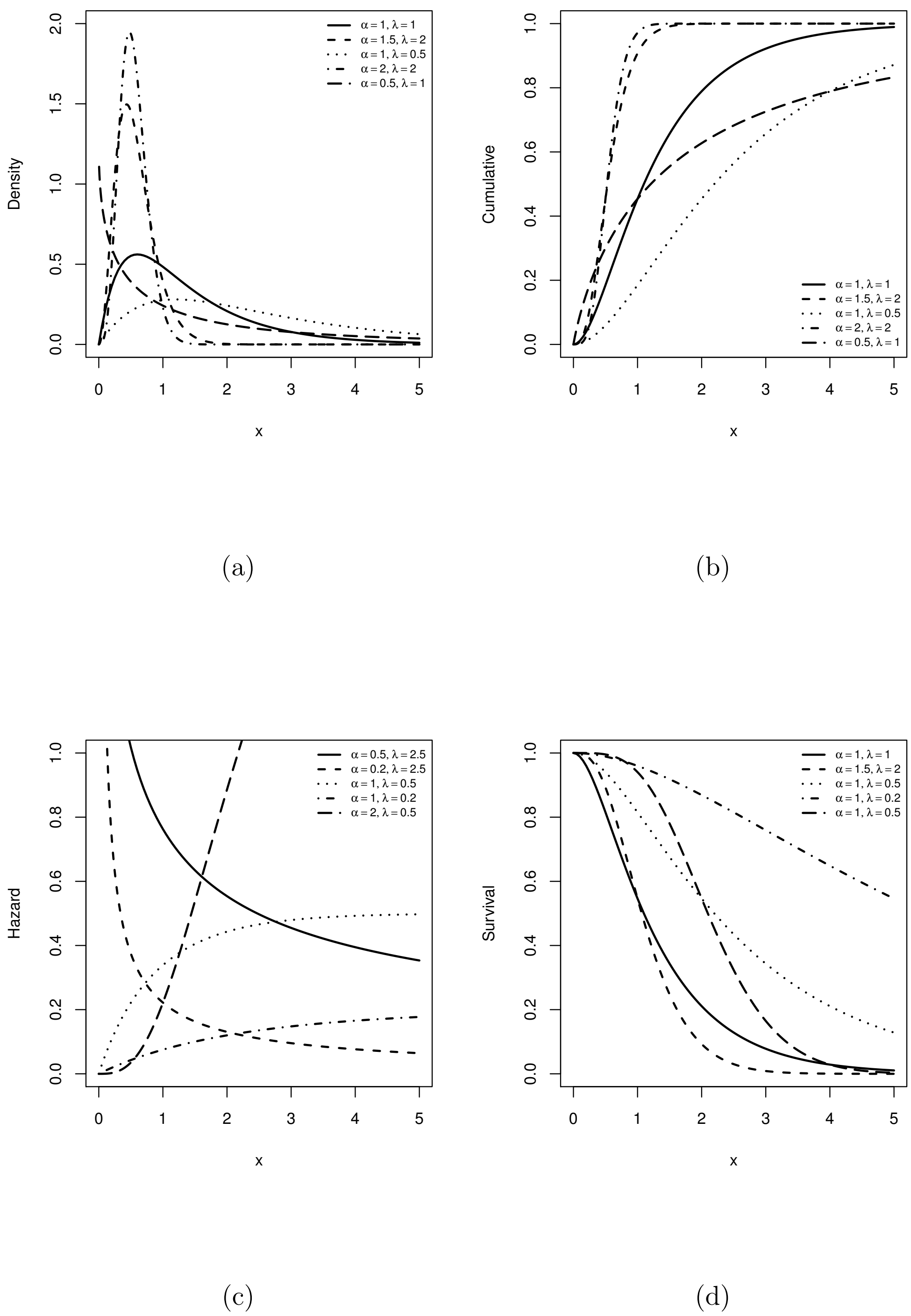

Figure 4.1: Plots of the (a) Cos-W density, (b) Cos-W cumulative, (c) Cos-W hazard rate function and (d) Cos-W survival

4.6 Expansion of the cumulative distribution and density of the Cos-W

In the following sections, we use the expansions defined for the **Cos-G** class of distributions in the development of the **Cos-W** distribution. Considering that $G(x)$ is the W distribution, using an identical manipulation as the one used for the class, we get the following expansion for the **Cos-W** cdf.

Theorem 4.6.1. *If $X \sim \mathbf{Cos} - \mathbf{W}(x; 2(k+1), \alpha, \lambda)$, then the cdf of the distribution can be expressed in terms of a linear combination of the cdf exponentiated Weibull distributions.*

$$H_W(x) = \sum_{k=0}^{\infty} b_{k+1} G_{2(k+1)}(x)\cdot$$

Where $b_{k+1} = \dfrac{(-1)^{k+1}}{(2k+2)!}\left(\dfrac{\pi}{2}\right)^{2k+2}$ and $G_{2(k+1)}(x)$ denotes the Weibull cumulative distribution with parameters $2(k+1), \alpha$ and λ.

Proof. We have just to integrate the pdf $h_W(x)$ from the Theorem 4.6.2 below. □

Theorem 4.6.2. *The pdf can be rewritten as an infinite linear combination of exponentiated Weibull densities.*

$$h_W(x) = \sum_{k=0}^{\infty} \omega_k g_{(j+1)}(x), \tag{4.11}$$

where $\omega_k = \displaystyle\sum_{j=0}^{2k+2} \left(\frac{\pi}{2}\right)^{2k+2} \frac{(-1)^{k+j+2}}{\Gamma(2k+2-j)j!(j+1)}$ and $g_{(j+1)}(x)$ denotes the Weibull density with parameters $(j+1), \alpha$ and λ.

Proof. First, if $|z| < 1$ and $b > 0$ is a nonnegative integer, we have representation

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} z^j, \tag{4.12}$$

using the power series (4.12), we can write Eq. (4.10) as

$$\begin{aligned}
h_W(x) &= \sum_{k=0}^{\infty} b_{k+1} 2(k+1)\alpha\lambda^{\alpha} x^{\alpha-1} \exp\left[-(\lambda x)^{\alpha}\right]\left\{1 - exp\left[-(\lambda x)^{\alpha}\right]\right\}^{2(k+1)-1} \\
h_W(x) &= \sum_{k=0}^{\infty}\sum_{j=0}^{2k+2} \left(\frac{\pi}{2}\right)^{2k+2} \frac{(-1)^{k+1}}{(2k+2)!}\frac{(-1)^j\Gamma(2k+2)}{\Gamma(2k+2-j)j!}(2k+2)\alpha\lambda^{\alpha}x^{\alpha-1}\exp\left[-(j+1)(\lambda x)^{\alpha}\right] \\
h_W(x) &= \sum_{k=0}^{\infty}\sum_{j=0}^{2k+2} \left(\frac{\pi}{2}\right)^{2k+2} \frac{(-1)^{k+j+2}(j+1)}{\Gamma(2k+2-j)j!(j+1)}\alpha\lambda^{\alpha}x^{\alpha-1}\exp\left[-(j+1)(\lambda x)^{\alpha}\right],
\end{aligned}$$

and we can rewrite Eq. (4.11) as a linear combination of Weibull densities.

$$h_W(x) = \sum_{k=0}^{\infty} \omega_k g_{(j+1)}(x;\alpha,\lambda), \tag{4.13}$$

where $\omega_k = \sum_{j=0}^{2k+1} \left(\frac{\pi}{2}\right)^{2k+2} \frac{(-1)^{k+j+2}}{\Gamma(2k+2-j)j!(j+1)}$ and $g_{(j+1)}(x;\alpha,\lambda)$ denotes the Weibull density with parameters $(j+1), \alpha$ and λ. □

4.6.1 Moment of order m for Cos-W distribution

In this section we introduce the expansion of order m for **Cos-W** distribution, as seen in the following situation:

$$\mu_m = E(X^m) \int_0^{+\infty} x^m dH(x),$$

by using the fdp expansion (4.4), we obtain

$$\mu_m = \sum_{k=0}^{\infty} \omega_k \int_0^{+\infty} x^m g_{(j+1)}(x;\alpha,\lambda)dx\cdot$$

by using $\omega_k = \sum_{j=0}^{2k+2} \left(\frac{\pi}{2}\right)^{2k+2} \frac{(-1)^{k+j+2}}{\Gamma(2k+2-j)j!(j+1)}$ and $y = (j+1)(\lambda x)^\alpha$ we have

$$\mu_m = \sum_{k=0}^{\infty} \frac{\omega_k}{(j+1)^{\frac{m}{\alpha}} \lambda^m} \int_0^{\infty} y^{\frac{m}{\alpha}} \exp(-y)dy,$$

finally, Eq. (4.13) can be rewritten as

$$\mu_m = \sum_{k=0}^{\infty} \frac{\omega_k}{(j+1)^{\frac{m}{\alpha}} \lambda^m} \Gamma\left(\frac{m}{\alpha}+1\right)\cdot$$

By using an approach similar to the one used to obtain the expansion of the moment generating function

$$M_X(t) \quad = \quad \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\omega_k}{(j+1)^{\frac{m}{\alpha}} \lambda^m} \frac{t^m}{m!} \int_0^{\infty} y^{\frac{m}{\alpha}} \exp(-y)dy$$

using $\upsilon_k(t) = \sum_{m=0}^{k} \frac{\omega_k}{(j+1)^{\frac{m}{\alpha}} \lambda^m} \frac{t^m}{m!}$

$$M_X(t) = \sum_{k=0}^{\infty} \upsilon_k(t) \Gamma\left(\frac{m}{\alpha}+1\right)\cdot$$

Using identical manipulation $\upsilon_k(it) = \sum_{m=0}^{k} \frac{\omega_k}{(j+1)^{\frac{m}{\alpha}}\lambda^m} \frac{i^m t^m}{m!}$

$$\varphi_X(t) = \sum_{k=0}^{\infty} \upsilon_k(it)\Gamma\left(\frac{m}{\alpha}+1\right)\cdot$$

4.6.2 Expansion to the central moments of order m for Cos-W distribution

By using an identical manipulation to the one used in the class before, we have the following:

$$\mu'_m = \sum_{k=0}^{\infty}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{m-r}{\alpha}}\lambda^{m-r}}\Gamma\left(\frac{m-r}{\alpha}+1\right)\cdot \tag{4.14}$$

Considering $m = 2$, we get the expansion for the variance of the distribution.

$$\sigma^2 = \mu'_2 = \sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{2-r}{\alpha}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{\alpha}+1\right)\cdot \tag{4.15}$$

4.6.3 Expansion to the general coefficient of the Cos-W distribution

Using an identical manipulation as the one used in (4.11), the general coefficient of the **Cos-W** is given by

$$C_g(m) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{m-r}{\alpha}}\lambda^{m-r}}\Gamma\left(\frac{m-r}{\alpha}+1\right)}{\left(\sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{2-r}{\alpha}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{\alpha}+1\right)\right)^{m/2}}.$$

The asymmetry and kurtosis (4.11) and (4.12) can be expressed, respectively, by

$$C_g(3) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{3}\binom{3}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{3-r}{\alpha}}\lambda^{3-r}}\Gamma\left(\frac{3-r}{\alpha}+1\right)}{\left(\sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{2-r}{\alpha}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{\alpha}+1\right)\right)^{3/2}},$$

and

$$C_g(4) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{4}\binom{4}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{4-r}{\alpha}}\lambda^{4-r}}\Gamma\left(\frac{4-r}{\alpha}+1\right)}{\left(\sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r \frac{\omega_k}{(j+1)^{\frac{2-r}{\alpha}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{\alpha}+1\right)\right)^{2}}.$$

4.6.4 Rényi entropy for Cos-W

Distribution Based on the manipulation used in (4.13) and pdf (4.2), the Rényi entropy [6] is given by

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma} \log \left[\int_0^{+\infty} \left(\frac{\alpha\lambda^\alpha \pi}{2} \right)^\gamma x^{(\alpha-1)\gamma} exp[-\gamma(\lambda x)^\alpha] \cos^\gamma \left[\frac{\pi}{2} exp(-(\lambda x)^\alpha) \right] \right] dx \cdot$$

By using Taylor series we can write where $W(s) = \cos^\gamma\left(\frac{\pi}{2}s\right) = \sum_{k=0}^{\infty} a_k (1-s)^k = \sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k \binom{k}{r}(-1)^r s^r$ and hence

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma} \log \left[\int_0^{+\infty} \left(\frac{\alpha\lambda^\alpha \pi}{2} \right)^\gamma x^{(\alpha-1)\gamma} exp[-\gamma(\lambda x)^\alpha] \sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k (-1)^r \binom{k}{r} \exp[-k(\lambda x)^\alpha] \right] dx \quad (4.16)$$

where $a_k = \dfrac{(-1)^k W^{(k)}(1)}{k!} \cdot$

Setting $y = (\gamma + k)\lambda^\alpha x^\alpha$ in Eq. (4.16), we have

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma} \times \log \left[\left(\frac{\alpha\lambda^\alpha \pi}{2} \right)^\gamma \sum_{k=0}^{\infty}\sum_{r=0}^{k} \binom{K}{r} \frac{(-1)^r a_k}{[(\gamma+k)\lambda^\alpha]^{\frac{(\alpha-1)(\gamma-1)}{\alpha}+1}} \Gamma\left[\frac{(\alpha-1)(\gamma-1)}{\alpha} + 1 \right] \right] \cdot$$

4.7 Maximum likelihood estimation

Calculating the estimation by the method of maximum likelihood estimator (MLE) for parameters α and λ. Where $\mathbf{x} = \{x_1, \cdots, x_n\}^\top$ of size n of independent random variables from the $\mathbf{Cos} - \mathbf{W}$. The log-likelihood and the score function are as seen

$$L = n \log\left(\frac{\pi}{2}\right) + n\log(\alpha\lambda^\alpha) + (\alpha-1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}(-(\lambda x_i)^\alpha) + \sum_{i=1}^{n}\log(1-\exp((-(\lambda x_i)^\alpha))) + \sum_{i=1}^{n}\log\left(\cos\left[\frac{\pi}{2}(1-\exp(-(\lambda x_i)^\alpha))\right]\right) \cdot$$

The elements of the score vector are given by

$$U_\lambda = \frac{n\alpha}{\lambda} - \frac{\alpha\pi}{2\lambda}\sum_{i=1}^{n}\tan\left[\left(\frac{\pi}{2}\right)(1-\exp(-(\lambda x)^\alpha)\right](\lambda x_i)^\alpha\exp(-(\lambda x_i)^\alpha)$$

and

$$U_\alpha = \frac{n\left(\lambda^\alpha + \alpha\lambda^\alpha\log(\lambda)\right)}{\alpha\lambda^\alpha} + \sum_{i=1}^{n}\log(x_i) - \frac{\pi}{2}\sum_{i=1}^{n}\tan\left[\frac{\pi}{2}(1-\exp(-(\lambda x_i)^\alpha))\right]$$
$$\times\left((\lambda x_i)^\alpha\right)\log(\lambda x_i)\left(\exp(-\left(\lambda x_i\right)^\alpha)\right)\cdot$$

4.8 Application

Here we use the **Cos-W** distribution in an application to a real data set. We will compare it with the Exponentiated Exponential - EE - Eq. (4.17) and the Weibull Eq. (4.18). The density functions is given by

$$f(x) = \alpha\lambda\exp(-\lambda x)\left[1-\exp(-\lambda x)\right]^{\alpha-1}, \tag{4.17}$$

and

$$f(x) = \alpha\lambda^\alpha x^{\alpha-1}\exp\left(-(\lambda x)^\alpha\right), x > 0\cdot \tag{4.18}$$

For the numerical analysis, was used R [5]. The data set presented contains 64 observations of time, in seconds, between consecutive eruptions of the geyser Kiama (Kiama Blowhole). Values were observed on July 12, 1998 with the aid of a digital clock by Professor Jim Irish (University of Technology, Sydney). Data were obtained on Smyth [8]. The data come from the windshield protector, including 64 classified observations as failure time saver windshield.

In Table 4.3 we have descriptive data and Table 4.4 brings the maximum likelihood estimates of the proposed model as well as the following statistics: AIC, BIC Akaike [1], and statistics Anderson-Darling [3] (A^*) to check how the distribution fits the data, and Cramér - von Mises (W^*). The smaller the values of these statistics, the better the model fit. We can observe, in Table 4.4, the results involving the Exponential exponentiated (EE) and Weibull

Table 4.2: Kiama Blowhole

83	51	87	60	28	95	8	27	28	56	8	25	68	146	89	18
73	69	9	37	10	82	29	8	60	61	61	18	169	25	8	26
11	83	11	42	17	14	9	12	15	10	18	16	29	54	91	8
17	55	10	35	47	77	36	17	21	36	18	40	10	7	34	27

Table 4.3: Descriptive statistics.

Min.	Q_1	Median	Mean	Q_3	Max.	Var.
7.00	14.75	28.00	39.83	60.00	169.00	1139.097

(W) distribution, and that the new model beats the Weibull model obtaining better statistics. Thus, we can say that the **Cos-W** distribution can be applied to explain the physical behavior of the geyser. We can observe that the Figure 4.2 an excellent fit to the data distribution to the adequacy of the data.

Table 4.4: MLE of the parameters of the Cos-W, W and EE models with error in parentheses and AIC, BIC, CAIC and HQIC statistics

Distributions	Estimates		AIC	BIC	CAIC	HQIC	W^*	A^*
Cos-W (α, λ)	0.92	0.04	594.35	598.68	594.55	596.06	0.1196	0.8516
	(0.0838)	(0.0041)						
W (α, λ)	1.27	0.02	597.80	602.11	597.99	599.50	0.1471	1.0079
	(0.1202)	(0.0023)						
EE (α, λ)	1.72	0.03	595.33	599.66	595.53	597.03	0.1287	0.9010
	(0.3158)	(0.0051)						

We can see that the new distribution provided a best fit according to all tests. Thus, we conclude that this distribution is quite flexible in the modeling of the proposed data.

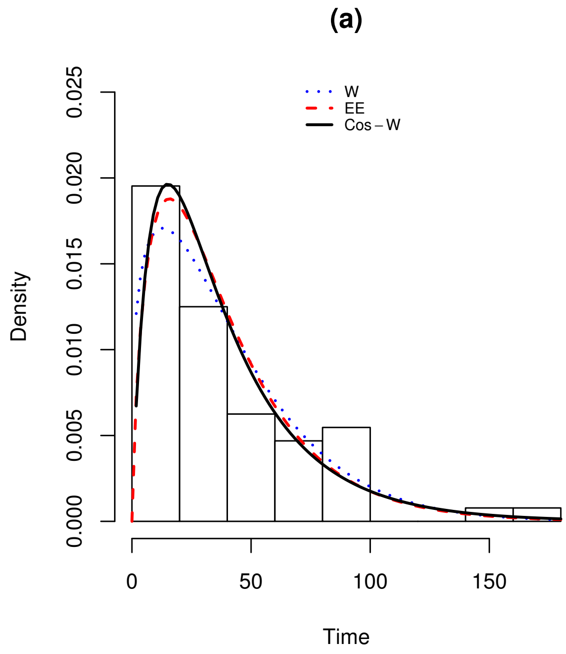

Figure 4.2: Some fitted densities of the data.

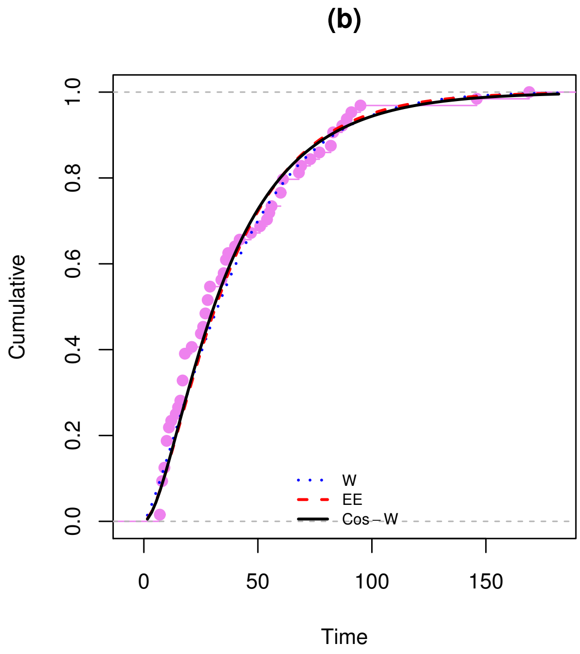

Figure 4.3: Some fitted cumulative of the data.

4.9 Concluding remarks

We developed the new **Cos-G** class of trigonometric distributions. Within this class we investigated a new distribution, the **Cos-W**. We obtained the density function, cumulative function and its expansions. We also calculated the entropy and their estimates via maximum likelihood. We hope that this model can help in the analysis of survival data, as well as in other areas of knowledge. We are working with others in the **Cos-W** distribution class in applications on real data about uterine cancer, as well as the use of entropy for this class.

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CHAPTER 5

New Classes of Trigonometric Distributions III: **Tan-G**

Resumo

Introduzimos uma nova classe de distribuições trigonométricas, chamada de **Tan-G** e analisamos um caso especial desta classe: a distribuição tangente Burr XII (**Tan-BXII**), que possui três parâmetros. Tratamento matemático da classe foi realizado, incluindo expansões para a função de densidade, momentos e entropia de Rényi. Obtivemos as estimativas sem forma fechada por máxima verossimilhança e a matriz de informação observada também foi obtida. Usamos um problema de falha em equipamentos como exemplo de aplicação para ilustrar a flexibilidade da nova distribuição.

Palavras-chaves: Classes trigonométricas tangente, Distribuição Burr XII; Estimação por máxima verossimilhança, Entropia.

Abstract

A new class of trigonometric distributions is introduced, **Tan-G**, and a special case of this class is analyzed: the Burr XII tangent distribution (**Tan-BXII**) which has three parameters. A Mathematical procedure of the class was carried out, including expansions for the density function, moments and Rényi entropy. The estimates were acquired in non-closed form by maximum likelihood estimation and the observed information matrix was also obtained. An equipment failure problem was used as an example of an application in order to illustrate the new distribution flexibility.

Palavras-chaves: Trigonometric tangent classes, Burr XII Distribution; Maximum likelihood estimation, Entropy.

5.1 Introduction

Recently, Brito [1] developed techniques to construct new classes of univariate and multivariate distributions. We use a baseline proposal for the construction of new classes of trigonometric probabilistic distributions. For any continuous baseline cumulative distribution function (cdf) $G(x)$, and $x > 0$ define

$$H_G(x) \quad = \quad \int_0^{\frac{\pi}{4}G(x)} \sec^2(t)dt, \tag{5.1}$$

can be rewritten (5.1) as

$$H_G(x) = \tan\left(\frac{\pi}{4}\ G(x)\right),$$

and if $G(x)$ has pdf $g(x)$ the pdf of the class is given by

$$h_G(x) = \frac{\pi}{4} g(x) \sec^2\left(\frac{\pi}{4} G(x)\right)\cdot$$

from now, we represent the class by **Tan-G**. If $G(x)$ is absolutely continuous, then its Hazard function is given by

$$R_G(x) = \frac{h_G(x)}{1 - H_G(x)}\cdot$$

The hrf of the **Tan-G** is given by

$$R_G(x) = \frac{\frac{\pi}{4} g(x) \sec^2\left(\frac{\pi}{4} G(x)\right)}{1 - \tan\left(\frac{\pi}{4}\ G(x)\right)}\cdot \tag{5.2}$$

5.1.1 Quantile function

Some of the properties of distribution can be studied through its moments, quantiles, skewness and kurtosis. Also, quantiles can be utilized to obtain data of the distribution according.

$$x = Q(u) = F^{-1}(u) = G^{-1}\left[\frac{4}{\pi}\arctan(u)\right]$$

Table 5.1: Quantile and random number generator

Algorithm Random generator for the **Tan-G** class
1. Generate $u \sim U(0,1)$.
2. Specify $G^{-1}(\cdot)$
3. Obtain an outcome of X by $X = Q(u)$

5.2 Useful expansion

5.2.1 Cumulative expansion distribution and Tan-G density class

Theorem 5.2.1. *If $G(x)$ is the cdf baseline of a random variable and $g(x)$ is its pdf, then the pdf of the class can be expressed in terms of the a linear combination of the pdf of exponentiated G distributions.*

$$h_G(x) = \sum_{k=1}^{\infty} \omega_k g_{(2k-1)}(x), \tag{5.3}$$

where $\omega_k = \left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}$ *and* $g_{(2k-1)}(x)$ *is exp-G.*

Proof. Using the Taylor series for the function, we have

$$\tan\left(\frac{\pi}{4}G(x)\right) = \sum_{k=1}^{\infty} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}\left(\frac{\pi}{4}G(x)\right)^{2k-1}$$

where B_k are the Bernoulli numbers. Thus, we can obtain the expansion for $H_G(x)$

$$H_G(x) = \sum_{k=1}^{\infty} \left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!} G^{2k-1}(x).$$

Similarly, we can obtain the expansion for $h_G(x)$, as shown below

$$\begin{aligned} h_G(x) &= \sum_{k=1}^{\infty}\left(\frac{\pi}{4}\right)^{2k-1}\frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}(2k-1)g(x)G^{2k-2}, \\ h_G(x) &= \sum_{k=1}^{\infty}\omega_k g_{(2k-1)}(x)\cdot \end{aligned} \tag{5.4}$$

Where $\omega_k = \left(\frac{\pi}{4}\right)^{2k-1}\frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}$ and $g_{(2k-1)}(x)$ denotes the pdf of the exponentiated $G(x)$ distribution. Clearly $\sum_{k=1}^{\infty}\omega_k = 1$. □

5.2.2 Moments of order m for the Tan-G class

We obtain the moment's expansion of order m for the Class as a linear combination of exponentiated densities, as in the following situation:

Theorem 5.2.2. *The moment's order m can be written as linear combination of the moment's order m of the exponentiated densities of $g(x)$ distribution*

$$\mu'_m = \sum_{k=1}^{\infty}\omega_k\mu_m^k\cdot$$

Proof.

$$\mu_m = E(X^m) = \int_0^{+\infty} x^m dH_G(x)\cdot$$

using the expansion for pdf

$$\mu_m = \int_0^{+\infty} x^m \sum_{k=1}^{\infty}\omega_k g_{2k-1}(x)dx,$$

which is equivalent to

$$\mu_m = \sum_{k=1}^{\infty}\omega_k \int_0^{+\infty} x^m g_{2k-1}(x)dx\cdot$$

Finally, we can rewrite the expansion as

$$\mu_m = \sum_{k=1}^{\infty}\omega_k\mu_m^k, \tag{5.5}$$

where ω_k is defined in Eq. (5.4), and μ_m^k is the m-th moment of the exponentiated $G(x)$ distribution. □

5.2.3 Central moments of order m for the Tan-G class distribution

Similarly, we can obtain the expansion for the central moments of order m, according to Theorem (5.2.2)

Corollary 5.2.1. *The central moments of order m is given by*

$$\mu'_m = \sum_{k=1}^{\infty} \gamma_m(k) \mu_m'^{(k)} \cdot$$

Where $\gamma_m(k) = \sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \sum_{k=1}^{\infty} \left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}$ *and* $\mu_m'^{(k)}$ *is the m-th central moments. Considering* $m = 2$ *we have the variance as:*

$$\sigma^2 = \mu'_2 \;\; = \;\; \sum_{k=1}^{\infty} \gamma_2(k) \mu_2'^{k} \cdot$$

5.2.4 Moment generating and characteristic function class

Theorem 5.2.3. *The moment generating function can be expressed as an infinite linear combination of the generating function moments of exponentiated* $G(x)$ *distribution.*

$$M_X(t) = \sum_{k=1}^{\infty} \upsilon_k(t) M_{Y_{(2k-1)}}(t) \cdot$$

Proof. Let $g_{2k-1}(x)$ the pdf exp-G, thus

$$M_X(t) = E(e^{tX}) = \int_0^{+\infty} e^{tx} dH_G(x),$$

using the expansion for pdf of the **Tan-G** class we can obtain

$$M_X(t) = \int_0^{+\infty} e^{tx} \sum_{k=1}^{\infty} \left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!} (2k-1) g(x) G^{2k-2}(x) dx,$$

which is equivalent to

$$M_X(t) = \sum_{k=1}^{\infty} \left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!} \int_0^{+\infty} e^{tx} (2k-1) g(x) G^{2k-2}(x) dx \cdot$$

Using Taylor series expansion of the exponential function

$$e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!},$$

we get

$$M_X(t)=\sum_{k=1}^{\infty}\left(\frac{\pi}{4}\right)^{2k-1}\frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}\int_0^{+\infty}\sum_{m=0}^{\infty}\frac{t^m x^m}{m!}(2k-1)g(x)G^{2k-2}(x)dx, \qquad (5.6)$$

finally, we can obtain

$$M_X(t)=\sum_{k=1}^{\infty}\upsilon_k(t)M_Y(t)\cdot \qquad (5.7)$$

Where $\upsilon_k(t)=\sum_{m=0}^{\infty}\left(\frac{\pi}{4}\right)^{2k-1}\frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}$ and $M_Y(t)$ is the moment generating function. Using an approach similar to the one used for the expansion of the moment generating function.

$$\varphi X(t)=\sum_{k=1}^{\infty}\upsilon_k(it)\phi_Y(t)\cdot \qquad (5.8)$$

Where $\upsilon_k(it)=\sum_{m=0}^{\infty}\left(\frac{\pi}{4}\right)^{2k-1}\frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}$ and $\phi_Y(t)$ is the characteristic function. □

5.2.5 Expansion to the general coefficient for the Tan-G class

The general coefficient of a random variable X for the class is given by

$$C_g(m) = \frac{\mu'_m}{\sigma^m}$$

$$C_g(m) = \frac{\sum_{k=1}^{\infty}\gamma_k(m)\mu_m^{'(k)}}{\left[\sum_{k=1}^{\infty}\gamma_k(2)\mu_2^{'(k)}\right]^{m/2}}.$$

So, the asymmetry and kurtosis can be respectively expressed by

$$C_g(3)=\frac{\sum_{k=1}^{\infty}\gamma_k(3)\mu_3^{'(k)}}{\left[\sum_{k=1}^{\infty}\gamma_k(2)\mu_2^{'(k)}\right]^{3/2}},$$

and

$$C_g(4)=\frac{\sum_{k=1}^{\infty}\gamma_k(4)\mu_4^{'(k)}}{\left[\sum_{k=1}^{\infty}\gamma_k(2)\mu_2^{'(k)}\right]^{2}}.$$

5.3 Entropy for the Tan-G class

Entropy measures the uncertainty Shannon [5], the greater the entropy, the higher the disorder and the less likely it will be to observe a phenomenon; the lower the entropy, the lower its disorder and the higher the probability of observing a particular event. Then, we obtain Rényi entropy [4] given by

$$\mathfrak{L}_{R,G}(\eta) \quad = \quad \frac{1}{1-\gamma} \log \left[\int_{-\infty}^{+\infty} h_G^{\gamma}(x) \right] dx \cdot \tag{5.9}$$

By expanding $W(s) = \sec^{2\gamma}\left[\frac{\pi}{2}\, s\right]$ in Taylor series, we have

$$\sec^{2\gamma}\left[\frac{\pi}{2}\, s\right] = \left(\frac{\pi}{4}(1-y)\right) = \sum_{k=0}^{\infty} a_k (1-y)^k = \sum_{k=0}^{\infty}\sum_{r=0}^{k} \binom{k}{r} (-1)^j a_k y^r, \tag{5.10}$$

where $a_k = \frac{(-1)^k\, W^{(k)}(1)}{k!}$, and $W^{(k)}(1)$ denotes the kth derivative of $W(\cdot)$ evaluated at the point 1. Next, we derive expressions for the Rényi entropy for the Tan-G class. Due to the fact that the parameter γ is not in general a natural number, it is difficult to use Equation (4.3). Substituting (5.3) and (5.9) in to Eq. (5.11), and after some algebra, we obtain

$$\mathfrak{L}_{R,G}(\gamma) = \frac{\gamma}{1-\gamma} \left\{ \log(\pi/4)^{\gamma} + \log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k\, (-1)^r \binom{k}{r} \mathrm{I}_r \right] \right\}, \tag{5.11}$$

where I_r comes from baseline distribution as

$$\mathrm{I}_r = \int_{-\infty}^{+\infty} G^r(x)\, g^{\gamma}(x)\, \mathrm{dx} \cdot$$

The quantity I_r can be computed at least numerically. The Shannon entropy is given by $E\{-\log[f(X)]\}$. It is a special case of the Rényi entropy when $\gamma \uparrow 1$.

5.4 Maximum likelihood estimation and score for the Tan-G class

We consider the estimation of the class by the method of maximum likelihood. Let $\underset{\sim}{x} = (x_1, \ldots, x_n)^{\top}$ be a random sample observations from the **Tan-G** with parameter θ. Then, the log-likelihood (LL) function for the class is:

$$\ell(\underset{\sim}{\theta}) = n\log\frac{\pi}{4} + \sum_{i=1}^{n}\log\left(G(x_i|\underset{\sim}{\theta})\right) + 2\sum_{i=1}^{n}\log\left[\sec\left(\frac{\pi}{4}G(x_i|\underset{\sim}{\theta})\right)\right].$$

Obtaining a score function

$$U(\theta_j) = \frac{\partial\ell(\underset{\sim}{\theta})}{\partial\theta_j},$$

and a score function for class in θ_j, for $j = 1, \ldots, p$ and $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_p)$ and p is the number of parameters of the distribution $G(x|\underset{\sim}{\theta})$.

$$U(\theta_j) = \sum_{i=1}^{n}\frac{1}{g(x_i|\underset{\sim}{\theta})}\frac{\partial g(x_i|\underset{\sim}{\theta})}{\partial\theta_j} + \sum_{i=1}^{n}\frac{1}{G(x_i|\underset{\sim}{\theta})}\frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial\theta_j} + \tan\left(\frac{\pi}{2}G(x_i|\underset{\sim}{\theta})\right)\frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial\theta_j}.$$

5.5 The Tan-BXII distribution

Considering that $G(x)$ is the BXII distribution, the cdf of the **Tan-BXII** distribution, for $x > 0$, is given by

$$H_G(x) = \tan\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k}\right)\right\}.$$

The pdf corresponding to Eq. (5.3) for $x > 0$, is given by

$$h_G = \frac{\pi}{4}\left\{x^{c-1}cks^{-c}\left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k-1}\right\} \times \sec^2\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k}\right)\right\}. \tag{5.12}$$

We represent the distribution by **Tan-BXII**. Finally, the hrf corresponding to (5.2)

$$R_G = \frac{\pi}{4}\left\{x^{c-1}cks^{-c}\left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k-1}\right\} \times \sec^2\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k}\right)\right\} /1 - \tan\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k}\right)\right\}.$$

5.6 Shape characteristics of density, cumulative, Hazard and survival

Some possible shapes of the density for some parameter values are displayed in Figure 5.1 and 5.2. We presented hrf in Figure 5.3 and 5.4 and for some parameter values. The hrf

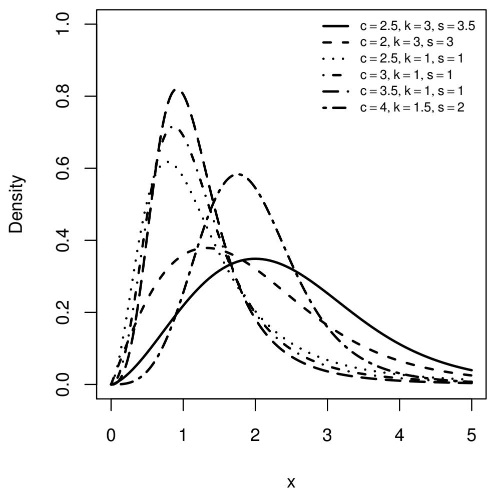


Figure 5.1: Plot of the Tan-BurrXII density

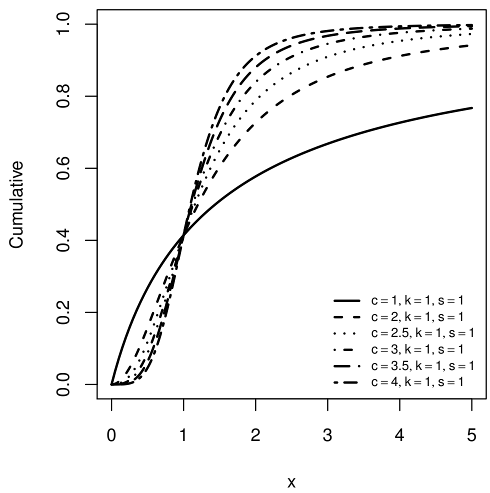


Figure 5.2: Plot of the Tan-BurrXII cumulative

can be unimodal or it can only be decreasing.

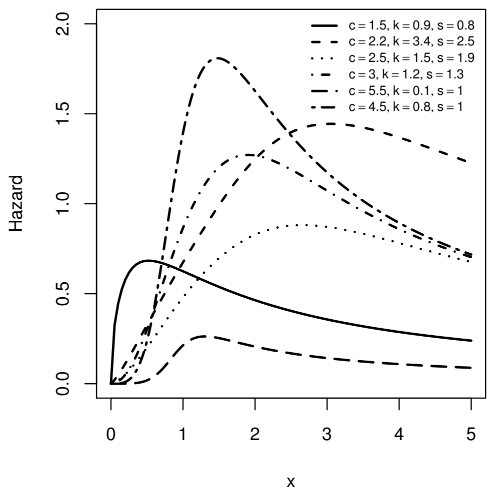


Figure 5.3: Tan-BurrXII Hazard unimodal

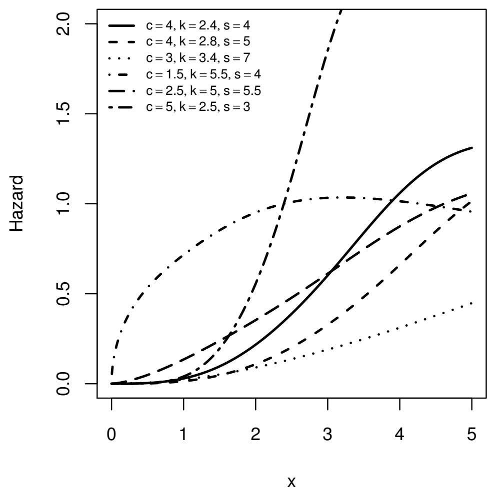


Figure 5.4: Tan-BurrXII Hazard increasing

5.7 Expansion of the cumulative distribution and density for Tan-BXII

In the following sections we use the expansions defined for the **Tan-G** class of distributions in the development of the **Tan-BXII** distribution. Considering that $G(x)$ is the BXII distribution, the cdf corresponding to equation (2.4), using an identical manipulation, is given by

Theorem 5.7.1. *If* $X \sim \mathbf{Tan-BXII}$, *then the density function is an infinite linear combination of BXII density function.*

$$h_{BurrXII}(x) = \sum_{k=1}^{\infty} \sum_{j=0}^{2k+1} \omega_{k,j} \; g_{BurrXII},$$

where $\omega_{k,j} = \left(\frac{\pi}{4}\right)^{2k+1} \frac{(-1)^{j+k} B_{2k+2} (4)^{k+1} (1-4^{k+1})}{\Gamma(2k+1-j)(j+1)!}$ *and* $g_{BurrXII}$ *is exp-BurrXII.*

Proof. Using Eq. (5.4), and

$$(1-z)^{\alpha} = \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)} z^j, \tag{5.13}$$

the pdf can be rewritten as

$$h_{BurrXII}(x) = \sum_{k=1}^{\infty}\left(\frac{\pi}{4}\right)^{2k+1} \frac{B_{2k+2}(-4)^{k+1}(1-4^{k+1})}{(2k+2)!} c\kappa s^{-c} x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-\kappa-1}$$
$$\times \sum_{j=0}^{2k+1}(-1)^j \frac{\Gamma(2k+1)}{j!\Gamma(2k+1-j)}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-j\kappa}.$$

using the binomial expansion in the last term

$$h_{BurrXII}(x) = \sum_{k=1}^{\infty}\left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^{k}(1-4^{k})}{2k(2k-2)!} c\kappa s^{-c} x^{c-1}$$
$$\times \sum_{j=0}^{2k+1}(-1)^j \frac{\Gamma(2k-1)}{j!\Gamma(2k-1-j)}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-\kappa(j+1)-1},$$

substituting in the last term, we obtain

$$h_{BurrXII}(x) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\left(\frac{\pi}{4}\right)^{2k+1} \frac{B_{2k+2}(4)^{k+1}(1-4^{k+1})}{(2k+2)!} c\kappa s^{-c} x^{c-1}$$
$$\times (-1)^{j+k} \frac{\Gamma(2k+1)}{j!\Gamma(2k+1-j)}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-\kappa(j+1)-1},$$

then,

$$h_{BurrXII}(x) = \sum_{k=1}^{\infty}\sum_{j=0}^{2k+1}\left(\frac{\pi}{4}\right)^{2k+1} \frac{(-1)^{j+k}B_{2k+2}(4)^{k+1}(1-4^{k+1})}{\Gamma(2k+1-j)j!} x^{c-1} c\kappa s^{-c}$$
$$\times \left[1+\left(\frac{x}{s}\right)^{c}\right]^{-\kappa(j+1)-1},$$

then, simplify

$$h_{BurrXII}(x) = \sum_{k=1}^{\infty}\sum_{j=0}^{2k+1}\omega_{k,j}(j+1)\; x^{c-1} c\kappa s^{-c}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-\kappa(j+1)-1},$$

reduces to

$$h_{BurrXII}(x) = \sum_{k=1}^{\infty}\sum_{j=0}^{2k+1}\omega_{k,j}\; burrXII(x|c,\kappa(j+1),s)\cdot \tag{5.14}$$

So, the equation above reveals that the $Tan-BXII$ density can be expressed as a linear combination of BXII where $g(x;c,(j+1)\kappa,s)$, where $\omega_{k,j} = \frac{(-1)^{j+k}B_{2k+2}(4)^{k+1}(1-4^{k+1})}{\Gamma(2k+1-j)(j+1)!}\left(\frac{\pi}{4}\right)^{2k+1}$ □

5.7.1 Moments of order m for Tan-BXII

Using an identical manipulation to the one used in Eq. (5.5), we introduce the moments' expansion

Theorem 5.7.2. *The moments of order m of* **Tan-BXII** *is*

$$\mu_m = \sum_{k=1}^{\infty}\sum_{j=0}^{2k} \omega_{k,j} s^m \kappa(j+1) B\left(\kappa(j+1) - mc^{-1}, 1 + mc^{-1}\right) \cdot$$

Proof.

$$\mu_m = \sum_{k=1}^{\infty}\sum_{j=0}^{2k} \omega_{k,j} \int_0^{\infty} (j+1) x^m x^{c-1} c\kappa s^{-c} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa(j+1)-1} dx,$$

be $u = \left(\frac{x}{s}\right)^c$,

$$\mu_m = \sum_{k=1}^{\infty}\sum_{j=0}^{2k} \omega_{k,j} (j+1) s^m \kappa \int_0^{\infty} u^{\frac{m}{c}} (1+u)^{-\kappa(j+1)-1} du,$$

with $\nu = (1+u)^{-1}$

$$\begin{aligned}
\mu_m &= \sum_{k=1}^{\infty}\sum_{j=0}^{2k} \omega_{k,j} s^m \kappa \int_0^{1} \nu^{\kappa(j+1) - \frac{m}{c} - 1} (1-\nu)^{\frac{m}{c}} d\nu \\
&= \sum_{k=1}^{\infty}\sum_{j=0}^{2k} \omega_{k,j} s^m \kappa(j+1) B\left(\kappa(j+1) - mc^{-1}, 1 + mc^{-1}\right) \cdot
\end{aligned}$$

□

5.7.2 Moment generating and characteristic functions for Tan-BXII

Using an identical manipulation to the one used in Eq. (5.6), the moments' function is given by

Theorem 5.7.3. *The moment generating function of the* **Tan-BXII** *is given by*

$$M_X(t) = \sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{m=0}^{\infty} \omega_{k,j} s^m \kappa(j+1) \frac{t^m}{m!} B\left(\kappa(j+1) - mc^{-1}, 1 + mc^{-1}\right) \cdot$$

Proof. Similar to the proof in Theorem (5.7.2) □

Using an identical manipulation in the equation below, the characteristic function is given by

$$\varphi_X(t) = \sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{m=0}^{\infty} \omega_{k,j} s^m \kappa(j+1) \frac{i^m t^m}{m!} B\left(\kappa(j+1) - mc^{-1}, 1 + mc^{-1}\right) \cdot$$

Where $\kappa(j+1) - mc^{-1} > 0$.

5.7.3 Central moments of order m for Tan-BXII

Using an identical manipulation in Corollary (5.2.1) in the case below, we get:

$$\mu'_m = \sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \omega_{k,j} s^{m-r} \kappa(j+1) B\left[\kappa(j+1) - (m-r)c^{-1}, 1 + (m-r)c^{-1}\right],$$

with $\kappa(j+1) - mc^{-1} > 0$. Considering $m = 2$, we get the expansion for variance of the distribution.

$$\sigma^2 = \mu'_2 = \sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{2} \binom{2}{r} (-1)^r \mu^r \omega_{k,j} s^{2-r} \kappa(j+1) B\left[\kappa(j+1) - (2-r)c^{-1}, 1 + (2-r)c^{-1}\right] \cdot$$

5.7.4 Expansion to the general rate for Tan-BXII

Using an identical manipulation to the one used in (**??**), we have:

$$C_g(m) = \frac{\sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \omega_{k,j} s^{m-r} \kappa(j+1) B\left[\kappa(j+1) - (m-r)c^{-1}, 1 + (m-r)c^{-1}\right]}{\left(\sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{2} \binom{2}{r} (-1)^r \mu^r \omega_{k,j} s^{2-r} \kappa(j+1) B\left[\kappa(j+1) - (2-r)c^{-1}, 1 + (2-r)c^{-1}\right]\right)^{m/2}},$$

so, the asymmetry and kurtosis can be respectively expressed by

$$C_g(3) = \frac{\sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{3} \binom{3}{r} (-1)^r \mu^r \omega_{k,j} s^{3-r} \kappa(j+1) B\left[\kappa(j+1) - (3-r)c^{-1}, 1 + (3-r)c^{-1}\right]}{\left(\sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{2} \binom{2}{r} (-1)^r \mu^r \omega_{k,j} s^{2-r} \kappa(j+1) B\left[\kappa(j+1) - (2-r)c^{-1}, 1 + (2-r)c^{-1}\right]\right)^{3/2}},$$

and

$$C_g(4) = \frac{\sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{4}\binom{4}{r}(-1)^r\mu^r\omega_{k,j}s^{4-r}\kappa(j+1)B\left[\kappa(j+1)-(4-r)c^{-1},1+(4-r)c^{-1}\right]}{\left(\sum_{k=1}^{\infty}\sum_{j=0}^{2k}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r\omega_{k,j}s^{2-r}\kappa(j+1)B\left[\kappa(j+1)-(2-r)c^{-1},1+(2-r)c^{-1}\right]\right)^2}.$$

5.8 Entropy for the Tan-BXII distributions

The entropy of a distribution is a measure of uncertainty; the greater the entropy, the higher the disorder and less likely to observe a given event. We obtain expansions for the Rényi [5] entropy defined by Eq. (5.9) and pdf Eq. (5.12) is given by

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^{+\infty}\left(\frac{\pi}{4}\right)^{\gamma}x^{(c-1)\gamma}c^{\gamma}\kappa^{\gamma}s^{-c\gamma}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-(\kappa+1)\gamma}\sec^{2\gamma}\left(\frac{\pi}{4}\left\{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-\kappa}\right\}\right)dx\right].$$

By using Taylor series we can write $W(s) = \sec^{2\gamma}\left(\frac{\pi}{4}(1-y)\right) = \sum_{k=0}^{\infty}a_k(1-y)^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^j a_k y^j$ and hence

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma}\log\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^j a_k\int_0^{+\infty}\left(\frac{\pi}{4}\right)^{\gamma}x^{(c-1)\gamma}c^{\gamma}\kappa^{\gamma}s^{-c\gamma}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-(\kappa+1)\gamma}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-j\kappa}dx\right] \quad (5.15)$$

where $a_k = \dfrac{(-1)^k W^{(k)}(1)}{k!}$.

Setting $u = \left(\frac{x}{s}\right)^c$ and $du = cx^{c-1}s^{-c}dx$, simplify $du = cu^{\frac{c-1}{c}s^{c-1}s^{-c}dx} = \dfrac{cu^{\frac{c-1}{c}}}{s}dx$ we have $dx = u^{\frac{1-c}{c}}\frac{s}{c}du$. Where $\mu = \dfrac{1}{1-\gamma}$, substituting in Eq. (5.15). Substituting $v = (1+u)^{-1}$ and after some algebraic, we have

$$\begin{aligned}\mathfrak{L}_{R,G}(\gamma) &= \frac{1}{1-\gamma}\log\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\pi}{4}\right)^{\gamma}(-1)^j a_k s^{\gamma+1}c^{\gamma-1}\kappa^{\gamma}\int_0^1(1-v)^{\frac{c-1}{c}\gamma}v^{-\left(\frac{c-1}{c}\gamma\right)+\kappa(\gamma+j)+\gamma-2}dv\right]\\ &= \frac{1}{1-\gamma}\log\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\pi}{4}\right)^{\gamma}(-1)^j a_k s^{\gamma+1}c^{\gamma-1}\kappa^{\gamma}B\left(\kappa(\gamma+j)+\gamma-\left(\frac{c-1}{c}\gamma\right)-1,\left(\frac{c-1}{c}\gamma\right)+1\right)\right].\end{aligned}$$

5.9 Maximum likelihood

Calculating the estimation by the method of maximum likelihood estimator (MLE) for parameters c, k and s, where $\mathbf{x} = \{x_1, \ldots, x_n\}^\top$ of independent random variables of size n from the $\mathbf{Tan-BurrXII}$, the log-likelihood and the score function are as presented below:

$$L = n\log\left(\frac{\pi}{4}\right) + (c-1)\sum_{i=1}^{n}\log(x_i) + n\log\left(cks^{-c}\right) - (k+1)\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{s}\right)^c\right)$$
$$+\sum_{i=1}^{n}\log\left[1-\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right] + \sum_{i=1}^{n}\log\left[\sec\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\right], \tag{5.16}$$

$$U_c = \sum_{i=1}^{n}\log(x_i) + \frac{n\left(ks^{-c} - cks^{-c}\log(s)\right)}{cks^{-c}} - \frac{(k+1)\sum_{i=1}^{n}\left(\frac{x_i}{s}\right)^c\log\left(\frac{x_i}{s}\right)}{1+\left(\frac{x_i}{s}\right)^c}$$
$$+\log\left\{-\frac{\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}(-k-1)\left(\frac{x_i}{s}\right)^c\log\left(\frac{x_i}{s}\right)}{1+\left(\frac{x_i}{s}\right)^c}\right\}$$
$$+\frac{(-k-1)\pi}{2}\sum_{i=1}^{n}\frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^c\log\left(\frac{x_i}{s}\right)}{1+\left(\frac{x_i}{s}\right)^c}, \tag{5.17}$$

$$U_k = \frac{n}{k} - \sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1} + \sum_{i=1}^{n}\log\left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\log\left(1+\left(\frac{x_i}{s}\right)^c\right)\right]$$
$$-\frac{\pi}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\log\left(1+\left(\frac{x_i}{s}\right)^c\right), \tag{5.18}$$

and

$$U_s = -\frac{nc}{s} + \frac{(k+1)\sum_{i=1}^{n}c\left(\frac{x_i}{s}\right)^c}{s\left(1+\left(\frac{x_i}{s}\right)^c\right)} + \sum_{i=1}^{n}\log\left(\frac{\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}(-k-1)\left(\frac{x_i}{s}\right)^c c}{s\left(1+\left(\frac{x_i}{s}\right)^c\right)}\right)$$
$$-\frac{(-k-1)\pi}{2}\sum_{i=1}^{n}\frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}(-k-1)\left(\frac{x_i}{s}\right)^c c}{s\left(1+\left(\frac{x_i}{s}\right)^c\right)}. \tag{5.19}$$

5.10 Application

Now we will apply the **Tan-BXII** distribution to model real data set and compare it with 3 other distributions Kum-BXII, BurrXII and Kum-W. These data are on the Aircraft windshield failures (thousands of hours) s reported in Murthy [3].

Table 5.2: Aircraft windshield failures (thousands of hours)

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610
3.478	0.557	1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914
2.646	3.699	1.124	1.981	2.661	3.779	1.248	2.010	2.688	3.924	1.281
2.038	2.823	4.035	1.281	2.085	2.890	4.121	1.303	2.089	2.902	4.167
1.432	2.097	2.934	4.240	1.480	2.135	2.962	4.255	1.505	2.154	2.964
4.278	1.506	2.190	3.000	4.305	1.568	2.194	3.103	4.376	1.615	2.223
3.114	4.449	1.619	2.224	3.117	4.485	1.652	2.229	3.166	4.570	1.652
2.300	3.344	4.602	1.757	2.324	3.376	4.663				

Table 5.3: Descriptive statistics.

Min.	Q_1	Median	Mean	Q_3	Max.	Var.
0.040	1.839	2.354	2.557	3.393	4.663	1.252

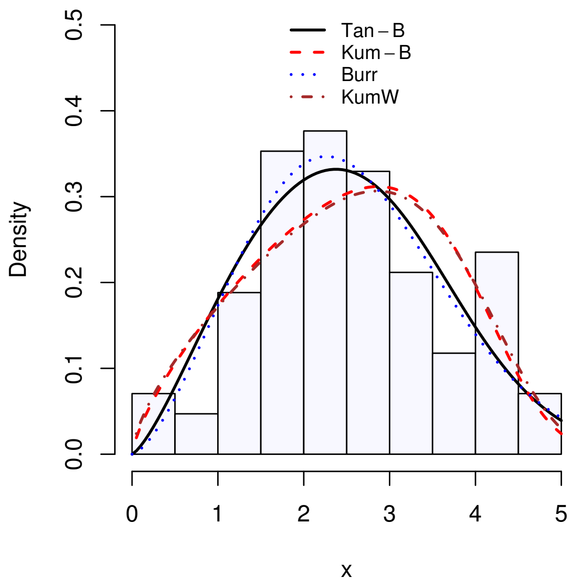
We can see that the new distribution, when compared to other ones, proved to have better statistics according to Anderson [2]. Thus, we conclude that this distribution is quite flexible in the modeling of the proposed data.

5.11 Concluding remarks

We develop a new **Tan-BXII** trigonometric distribution class in a novel class of trigonometric distributions, **Tan-G**. We obtain density function, cumulative function and its expansions. The entropy was also calculated and their estimates checked via maximum likelihood. We hope that this model can help in the analysis of survival data, as well as in

Table 5.4: MLE of the parameters of the Tan-BXII, Kum-BXII, BurrXII and Kum-W models with error in parentheses and AIC, BIC and CAIC statistics

Distributions	Estimates					AIC	BIC	CAIC	W^*	A^*
$Tan-BXII(c,\kappa,s)$	2.32	26.45	10.58	—	—	268.35	275.68	276.44	0.06	0.59
	(0.21)	(47.69)	(8.73)	—	—					
$Kum-BXII(a,b,c,d,k)$	0.24	1.22	7.19	5.62	5.32	267.85	280.07	268.61	0.11	0.77
	(0.07)	(1.14)	(2.37)	(10.62)	(1.69)					
$Kum-W(a,b,c,\beta)$	2.50	11.50	7.46	—	—	270.22	277.55	270.52	0.06	0.62
	(0.23)	(8.84)	(2.70)	—	—					
$BXII(a,c,k)$	0.32	0.96	5.22	0.26	—	265.66	275.44	266.16	0.11	0.73
	(0.26)	(0.61)	(3.88)	(0.08)	—					



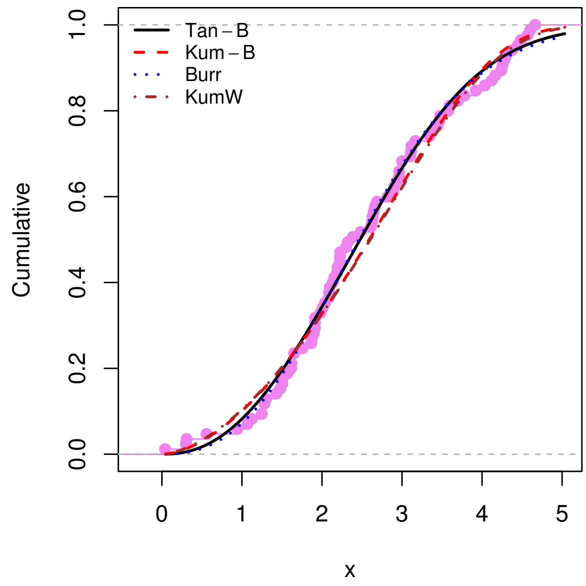


Figure 5.5: Some fitted densities of the data. Figure 5.6: Some fitted cumulative of the data.

other areas of knowledge. We can observe that the Figure 5.5 an excellent fit to the data distribution.

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CHAPTER 6

New Classes of Trigonometric Distributions IV: **Sec-G**

Resumo

Introduzimos uma nova classe de distribuições trigonométricas, chamada de **Sec-G**. Uma nova distribuição foi gerada da classe, a distribuição **Sec-KumW**, com apenas quatro parâmetros. Calculamos as expansões para a função de densidade e expansões para os momentos. O método de máxima verossimilhança é utilizado para estimar os parâmetros do modelo e, não possui forma fechada. Utilizamos a nova distribuição em uma aplicação dados reais.

Palavras-chaves: Classes trigonométricas secante, Distribuição Kumaraswamy Weibull; Estimação por máxima verossimilhança.

Abstract

A new class of trigonometric distributions is introduced, called **Sec-G**, and a new distribution came up from this class, the **Sec-KumW**, which has only four parameters. In addition to that, expansions for the density function, for the moment. The maximum likelihood method is used for estimating the parameters of the model, without any closed form. The new distribution was used in real-life data set application.

Keywords: Classes of trigonometric secant, Kumaraswamy Weibull distribution; Maximum likelihood estimation.

6.1 Introduction

Brito [3] developed techniques to construct new classes of distributions and univariate and multivariate distributions. We use a baseline proposal for the construction of new classes of trigonometric probabilistic distributions. The Kumaraswamy (Kum) has received attention in the literature. Studies of the main properties, such as failure rate have been developed. The novel cumulative distribution (cdf) proposed here is given by

$$H_G(x) = \int_0^{\frac{\pi}{3}\ G(x)} \sec(t)\tan(t)dt$$

If $G(x)$ baseline distribution is continuous, the cdf can be written as

$$H_G(x) \quad = \quad \sec\left(\frac{\pi}{3}\ G(x)\right) - 1, \tag{6.1}$$

and if $G(x)$ has pdf $g(x)$, the pdf of the class is given by

$$h_G(x) = \frac{\pi}{3} g(x) \sec\left(\frac{\pi}{3} G(x)\right) \tan\left(\frac{\pi}{3} G(x)\right) \cdot \tag{6.2}$$

We represent the class by **Sec-G** and the parameters of the new model will are the same of the probability distribution $G(x)$.

6.1.1 The Hazard function using the Sec-G

class If $G(x)$ is cdf absolutely continuous, then its hrf is given by

$$R_G(x) = \frac{h_G(x)}{1 - H_G(x)} \cdot \tag{6.3}$$

Using Eq.(6.1) and Eq.(6.2) in Eq.(6.3) we obtain the hazard function in the class as

$$R_G(x) = \frac{\frac{\pi}{3} g(x) \sec\left(\frac{\pi}{3} G(x)\right) \tan\left(\frac{\pi}{3} G(x)\right)}{2 - \sec\left(\frac{\pi}{3}\ G(x)\right)}.$$

6.1.2 Quantile function

Some of the properties of distribution can be studied through its moments, quantiles, skewness and kurtosis. Also, quantiles can be utilized to obtain data of the distribution according.

$$x = Q(u) = F^{-1}(u) = G^{-1}\left[\frac{3}{\pi} \operatorname{arcsec}(u+1)\right]$$

Table 6.1: Quantile and random number generator

Algorithm Random generator for the **Sec-G** class
1. Generate $u \sim U(0,1)$.
2. Specify $G^{-1}(\cdot)$
3. Obtain an outcome of X by $X = Q(u)$

6.2 Useful expansions

Theorem 6.2.1. *If $G(x)$ is the cdf baseline of a random variable and $g(x)$ is its pdf, then the pdf of the class can be expressed in terms of the a linear combination of the pdf of exponentiated G distributions.*

$$h_G(x) = \sum_{k=0}^{\infty} \omega_{k+1} g_{(2k+2)}(x),$$

where $\omega_{k+1} = \left(\frac{\pi}{3}\right)^{2k+2} \frac{(-1)^{k+1} E_{2k+2}}{(2k+2)!}$, E_{2k+2} *is the Euler number and* $g_{(2k+2)}(x)$ *is exp-G.*

Proof. Writing the function sec in terms of his expansion in Taylor series

$$\sec\left(\frac{\pi}{3} G(x)\right) = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!} \left(\frac{\pi}{3} G(x)\right)^{2k}$$

For arbitrary baseline cdf $G(x)$, a random variable is said to have the exponentiated-G ($exp-G$) with power parameter $a > 0$. Which gives the pdf as,

$$h_G(x) = \sum_{k=0}^{\infty} \left(\frac{\pi}{3}\right)^{2k} \frac{(-1)^k}{(2k)!}(2k)g(x)G^{(2k-1)}(x)\cdot$$

Here we have a problem, the first derivative, for $k = 0$ the first term of the expansion will be null, so we rewrite the new expansion as $G_{(2(k+1))}(x) = G^{(2k+2)}$ and $g_{(2k+2)}(x) = (2k+2)g(x)G^{(2k+1)}$ denotes the density of the exponentiated $G(x)$ distribution.

$$\begin{aligned} h_G(x) &= \sum_{k=0}^{\infty} \left(\frac{\pi}{3}\right)^{2k+2} \frac{(-1)^{k+1}E_{2k+2}}{(2k+2)!}(2k+2)g(x)G^{2k+1}(x), \\ h_G(x) &= \sum_{k=0}^{\infty} \omega_{k+1} g_{(2k+2)}(x)\cdot \end{aligned} \tag{6.4}$$

Where $\omega_{k+1} = \left(\frac{\pi}{3}\right)^{2k+2} \frac{(-1)^{k+1}E_{2k+2}}{(2k+2)!}$ and $g_{(2k+2)}(x)$ is a density of the exponentiated $G(x)$ distribution. □

Corollary 6.2.1. *If $h_G(x)$ denotes the density function, then, after integration, the cumulative distribution function is given by*

$$H_G(x) = \sum_{k=0}^{\infty} \omega_{k+1} G_{(2k+2)}(x)\cdot \tag{6.5}$$

6.3 General properties

The aim of this section is to investigate the mathematical properties of the class. Based on equations (6.4) and (6.5), in terms of infinite linear combination of exponentiated pdf and cdf, one obtains many results for the class. It follows directly from those ordinary moments properties, central moments, the moments generating function, characteristic function and entropy for the class expressed as an infinite linear combination.

6.3.1 Moments of order m for Sec-G class

Let Y be a random variable having the $h_G(x)$ pdf (6.4), then the moment expansion of order m for the **Sec-G** class is given by

Theorem 6.3.1. *If $Y_{2k+2} \sim G_{2k+2}(x)$ random variable and according Theorem (6.2.1), the moment's of order m can be rewritten as an infinite linear combination of the moment's of order m of exponentiated $G(x)$ densities.*

$$\mu_m = \sum_{k=0}^{\infty} \omega_{k+1} E\left[Y_{(2k+2)}^m\right]$$

Proof. Using the pdf expansion Eq. (6.4), we can write

$$\mu_m = E(X^m) = \int_0^{+\infty} x^m dG(x) \cdot \tag{6.6}$$

Now, by (6.6) we can write

$$\mu_m = \int_0^{+\infty} x^m \sum_{k=0}^{\infty} \omega_{k+1} g_{(2k+2)}(x) dx,$$

which is equivalent to

$$\begin{aligned}
\mu_m &= \sum_{k=0}^{\infty} \omega_{k+1} \int_0^{+\infty} x^m g_{(2k+2)}(x) dx \\
\mu_m &= \sum_{k=0}^{\infty} \omega_{k+1} E\left[Y_{(2k+2)}^m\right] \cdot
\end{aligned}$$

□

6.3.2 Moment generating and characteristic functions for Sec-G class

In this section, we derive a simple representation for the moment generating function mgf expansion.

Theorem 6.3.2. *If $Y_{2k+2} \sim G_{2k+2}(x)$, random variable, then we immediately obtain the moment generating function as an infinite linear combination of the moment generating function of exponentiated $G(x)$ densities.*

$$M_X(t) = \sum_{k=0}^{\infty} \upsilon_k M_Y(t)$$

Proof. using the pdf expansion (6.4), we obtain

$$M_X(t) = \int_0^{+\infty} e^{tx} \sum_{k=0}^{\infty} \omega_{k+1} g_{(2k+2)}(x) dx \tag{6.7}$$

which is equivalent to

$$M_X(t) = \sum_{k=0}^{\infty} \omega_{k+1} \int_0^{+\infty} e^{tx} g_{(2k+2)}(x)dx,$$

using the Maclaurin expansion series of Abramowitz exponential function [2],

$$e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!}, \tag{6.8}$$

we obtain

$$M_X(t) = \sum_{k=0}^{\infty} \omega_{k+1} \int_0^{+\infty} \sum_{m=0}^{\infty} \frac{x^m t^m m!}{g} {}_{(2k+2)}(x)dx, \tag{6.9}$$

finally, (6.9) can be rewritten as

$$\begin{aligned} M_X(t) &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \omega_{k+1} \int_0^{+\infty} \frac{x^m t^m}{m!} g_{(2k+2)}(x)dx, \\ M_X(t) &= \sum_{k=0}^{\infty} \upsilon_k M_Y(t) \cdot \end{aligned} \tag{6.10}$$

Where $\upsilon_k = \sum_{m=0}^{\infty} \omega_{k+1}$ and $M_Y(t)$ is mgf of the random variable with pdf (6.4). We can write the characteristic function as

$$\varphi_X(t) = \sum_{k=0}^{\infty} \phi_k \varphi_Y(t) \cdot$$

Where $\phi_k = \sum_{m=0}^{\infty} \omega_{k+1}$ and $\varphi_Y(t)$ is the characteristic function (chf) of the exponentiated $G(x)$ densities. □

6.3.3 Central moments of order m for Sec-G class

We now calculate the expansion for central moments of order m for the proposed class. By definition

Theorem 6.3.3. *Let $X \sim g_{2k+2}(x)$ be density function (6.4), then the central moments of order m is given by*

$$\mu'_m = \sum_{k=1}^{\infty} \gamma_k(m) \mu_m^k$$

where $\gamma_k(m) = \sum_{r=0}^{m} \binom{m}{r} \mu^r \left(\frac{\pi}{3}\right)^{2k+2} \frac{(-1)^{k+r+1} E_{2k+2}}{(2k+2)!}$.

For the $m = 2$, the variance can be expressed as:

$$\sigma^2 = \mu'_2 = \sum_{k=1}^{\infty} \gamma_k(2) \mu_2^k \cdot$$

6.3.4 General rate for Sec-G class

We, then, developed the expansion rate for the class as:

$$C_g(m) = \frac{\mu'_m}{\sigma^{m/2}}$$

$$C_g(m) = \frac{\sum_{k=1}^{\infty} \gamma_k(m) \mu_m'^k}{\left[\sum_{k=1}^{\infty} \gamma_k(2) \mu_2'^k\right]^{m/2}}.$$

So, the asymmetry and kurtosis can be respectively expressed by

$$C_g(3) = \frac{\sum_{k=1}^{\infty} \gamma_k(3) \mu_3'^k}{\left[\sum_{k=1}^{\infty} \gamma_k(2) \mu_2'^k\right]^{3/2}},$$

and

$$C_g(4) = \frac{\sum_{k=1}^{\infty} \gamma_k(4) \mu_4'^k}{\left[\sum_{k=1}^{\infty} \gamma_k(2) \mu_2'^k\right]^{2}}.$$

6.4 Maximum likelihood estimation and score for the Sec-G class

Let $\underset{\sim}{x} = (x_1, \ldots, x_n)^\top$ be a random variable of the **Sec-G** with the parameter θ. The log-likelihood (LL) functions for **Sec-G** are:

$$\ell(\underset{\sim}{\theta}) = n \log \frac{\pi}{3} + \sum_{i=1}^{n} \log\left(g(x_i|\underset{\sim}{\theta})\right) + \sum_{i=1}^{n} \log\left(G(x_i|\underset{\sim}{\theta})\right) + \sum_{i=1}^{n} \log\left(\sec\left(\frac{\pi}{3} G(x_i|\underset{\sim}{\theta})\right)\right) + \sum_{i=1}^{n} \log\left(\tan\left(\frac{\pi}{3} G(x_i|\underset{\sim}{\theta})\right)\right) \cdot \quad (6.11)$$

The score vector components $U(\theta)$ will be obtained through the partial derivatives of (6.11). We get the following

$$U(\theta_j) = \sum_{i=1}^{n} \frac{1}{g(x_i|\underset{\sim}{\theta})} \frac{\partial g(x_i|\underset{\sim}{\theta})}{\partial \theta_j} + \sum_{i=1}^{n} \frac{-1}{G(x_i|\underset{\sim}{\theta})} \frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial \theta_j}$$
$$- \sum_{i=1}^{n} \left(\frac{\pi}{3}\right) \tan\left(\frac{\pi}{3} G(x_i|\underset{\sim}{\theta})\right) \frac{\partial G(x_i|\underset{\sim}{\theta})}{\partial \theta_j}$$

where $j = 1, \ldots, p$ and $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_p)$ and p is the number of parameters of the distribution $G(x|\underset{\sim}{\theta})$.

6.5 The New Sec-KumW

Distribution Considering that $G(x)$ is the KumW distribution, the pdf corresponding to Eq. (6.2), for $x > 0$ is given by

$$h_G(x) = \frac{\pi}{3}\left(abc\lambda^c x^{(c-1)} \exp(-(\lambda x)^c)(1-\exp(-\lambda x)^c)^{(a-1)}(1-(1-\exp(-\lambda x)^c)^a)^{(b-1)}\right)$$
$$\times \sec\left(\frac{\pi}{3}(1-(1-(1-\exp(-(\lambda x)^c))^a)^b)\right) \times \tan\left(\frac{\pi}{3}(1-(1-(1-\exp(-(\lambda x)^c))^a)^b)\right).$$

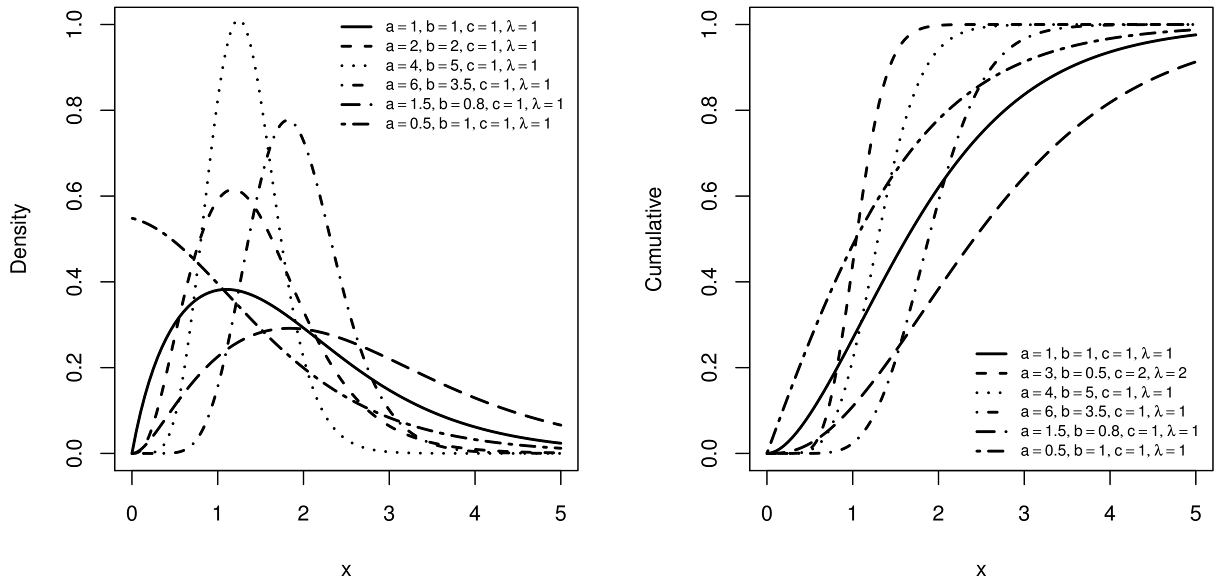
The cdf corresponding to Eq. (6.1) for $x > 0$, is given by

$$H_G(x) = \sec\left(\frac{\pi}{3}(1-(1-(1-\exp(-(\lambda x)^c))^a)^b)\right) - 1$$

from now, we represent the distribution **Sec-KumW**. Finally, the hrf is given by

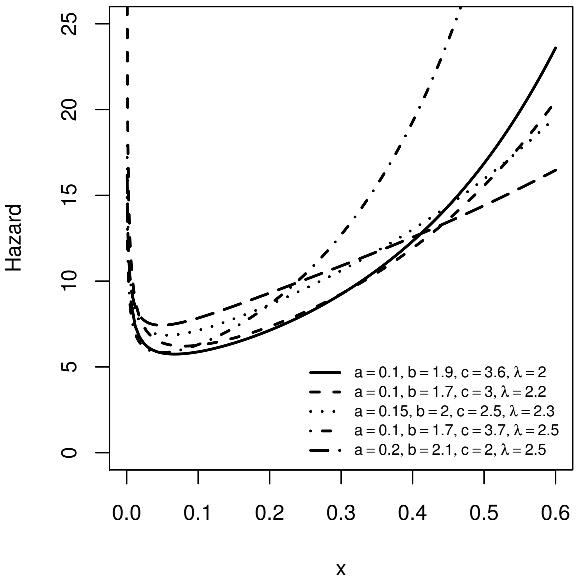
$$R_G(x) = \frac{\pi}{3}\left(abc\lambda^c x^{(c-1)} \exp(-(\lambda x)^c)(1-\exp(-\lambda x)^c)^{(a-1)}(1-(1-\exp(-\lambda x)^c)^a)^{(b-1)}\right)$$
$$\times \sec\left(\frac{\pi}{3}(1-(1-(1-\exp(-(\lambda x)^c))^a)^b)\right) \times \tan\left(\frac{\pi}{3}(1-(1-(1-\exp(-(\lambda x)^c))^a)^b)\right)$$
$$/\left(2 - \sec\left(\frac{\pi}{3}(1-(1-(1-\exp(-(\lambda x)^c))^a)^b)\right)\right)$$

Figure 6.1b illustrates this shape for some parameter values; plot of the density (6.1b) and cumulative marks (6.2). The distribution has a bathtub (6.2a). The probability distribution has unimodal (6.2b) form, which could be represented by an increasing and decreasing.

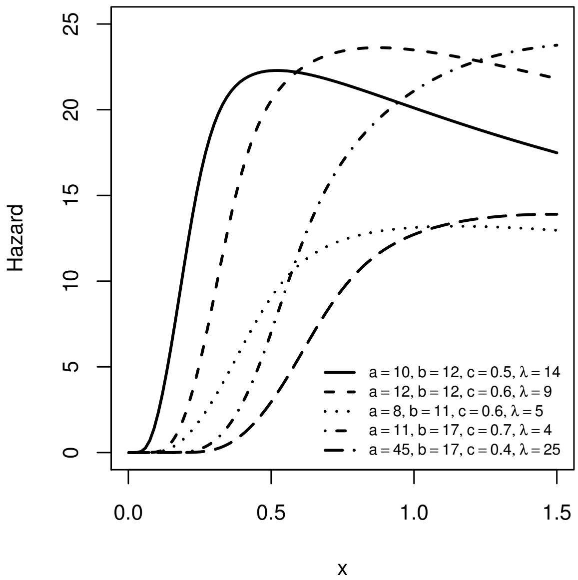


(a) Plot of the Sec-KumW density

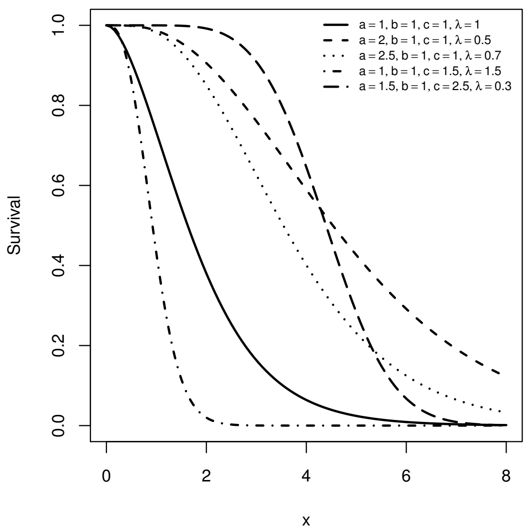
(b) Plot Sec-KumW cumulative



(a) Plot of the Sec-KumW Hazard bathtub



(b) Plot of the Sec-KumW Hazard unimodal



(c) Plot of Sec-KumW survival

Figure 6.2: Plot of the Sec-KumW Hazard bathtub, Sec-KumW hrf unimodal, Sec-KumW Hazard increasing and Sec-KumW survival

6.6 Useful expansion

In the following sections, we use the expansions defined for the **Sec-G** class in the development of the distribution **Sec-KumW**. Considering that $G(x)$ is the Kum-W distribution, the cdf corresponding to equation (6.5) is given by

Theorem 6.6.1. *If* $X \sim \mathbf{Sec-KumW}$, *then we can express the pdf of the* **Sec-KumW** *distribution as an infinite linear combination of Weibull distribution.*

$$h_G(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{2k+1}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty} \upsilon_{k,j,l,s}\, g_{(c,(s+1)^{\frac{1}{c}}\lambda)}(x)$$

where

$$\upsilon_{k,j,l,s} = ab\left(\frac{\pi}{3}\right)^{2k+2}\binom{2k+1}{j}\binom{b(j+1)-1}{l}\binom{a(l+1)-1}{s}\frac{(-1)^{k+j+l+s}E_{2k+2}}{(2k+1!)(s+1)} \quad (6.12)$$

and $g_{(c,(s+1)^{\frac{1}{c}}\lambda)}(x) = Weibull(x|c,(s+1)^{\frac{1}{c}}\lambda)$

Corollary 6.6.1. *The moment of order* m *of the* **Sec-KumW** *distribution is given by*

$$\mu_m = \sum_{k=0}^{\infty}\sum_{j=0}^{2k+1}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}}\lambda^m}\Gamma\left(\frac{m}{c}+1\right).$$

Where $\upsilon_{k,j,l,s}$ *is given by (6.12)*

Proof.

$$\mu_m = E(X^m)\int_0^{+\infty} x^m dH(x),$$

by using the fdp expansion, we obtain

$$\mu_m = \sum_{k=0}^{\infty}\sum_{j=0}^{2k+1}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty} \upsilon_{k,j,l,s}\int_0^{+\infty} x^m g_{(c,(s+1)^{\frac{1}{c}}\lambda)}(x)dx,$$

by using Eq. (6.12) and $y = c,(s+1)^{\frac{1}{c}}(\lambda x)^{\alpha}$ we have

$$\mu_m = \sum_{k=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}}\lambda^m}\int_0^{\infty} y^{\frac{m}{c}}\exp(-y)dy, \quad (6.13)$$

finally, Eq. (6.13) can be rewritten as

$$\mu_m = \sum_{k=0}^{\infty}\sum_{j=0}^{2k+1}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}}\lambda^m}\Gamma\left(\frac{m}{c}+1\right).$$

□

Corollary 6.6.2. *The moment generating function of the* **Sec-KumW** *distribution is given by*

$$M_X(t) = \sum_{k=0}^{\infty} \omega_k(t) \Gamma\left(\frac{m}{c} + 1\right) \cdot$$

Where $\omega_k(t) = \sum_{m=0}^{k} \sum_{j=0}^{2k+1} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}} \lambda^m}$.

Proof. By using an approach similar to the one used to obtain the expansion of the moment generating function

$$M_X(t) = \sum_{m=0}^{k} \sum_{j=0}^{2k+1} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}} \lambda^m} \int_0^{\infty} y^{\frac{m}{c}} \exp(-y) dy,$$

using $\upsilon_k(t) = \sum_{m=0}^{k} \sum_{j=0}^{2k+1} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}} \lambda^m}$

$$M_X(t) = \sum_{k=0}^{\infty} \upsilon_k(t) \Gamma\left(\frac{m}{c} + 1\right) \cdot$$

□

Corollary 6.6.3. *The characteristic function (chf) of the* **Sec-KumW** *distribution is given by*

$$\varphi_X(t) = \sum_{k=0}^{\infty} \omega_k(it) \Gamma\left(\frac{m}{c} + 1\right) \cdot$$

Where $\omega_k(it) = \sum_{m=0}^{k} \sum_{j=0}^{2k+1} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}} \lambda^m}$.

Proof. Using identical manipulation $\upsilon_k(it) = \sum_{m=0}^{k} \sum_{j=0}^{2k+1} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m}{c}} \lambda^m}$

$$\varphi_X(t) = \sum_{k=0}^{\infty} \upsilon_k(it) \Gamma\left(\frac{m}{c} + 1\right) \cdot$$

□

Corollary 6.6.4. *The central moment of order m of the* **Sec-KumW** *distribution is given by*

$$\mu'_m = \sum_{k=0}^{\infty} \sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m-r}{c}} \lambda^{m-r}} \Gamma\left(\frac{m-r}{c} + 1\right) \cdot$$

Considering $m = 2$, we get the variance expansion of the distribution.

$$\sigma^2 = \sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r\frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{2-r}{c}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{c}+1\right).$$

Proof.

$$\mu'_m = \sum_{k=1}^{\infty}\gamma_k(m)\mu_m^k$$

where $\gamma_k(m) = \sum_{r=0}^{m}\binom{m}{r}\mu^r\frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{2-r}{c}}\lambda^{2-r}}$ and after some algebra $\mu_m^k = \Gamma\left(\frac{2-r}{c}+1\right)$. □

6.6.1 Expansion to the general coefficient of the Sec-KumW distribution

We obtain the general coefficient of the **Sec-KumW** as depicted below

$$C_g(m) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\mu^r\frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m-r}{c}}\lambda^{m-r}}\Gamma\left(\frac{m-r}{c}+1\right)}{\left(\sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r\frac{\upsilon_{k,j,l,s}}{(j+1)^{\frac{2-r}{c}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{c}+1\right)\right)^{m/2}}.$$

The asymmetry and kurtosis are respectively given as:

$$C_g(3) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{3}\binom{3}{r}(-1)^r\mu^r\frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m-r}{c}}\lambda^{3-r}}\Gamma\left(\frac{3-r}{c}+1\right)}{\left(\sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r\frac{\upsilon_{k,j,l,s}}{(j+1)^{\frac{2-r}{c}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{c}+1\right)\right)^{3/2}},$$

and

$$C_g(4) = \frac{\sum_{k=0}^{\infty}\sum_{r=0}^{4}\binom{4}{r}(-1)^r\mu^r\frac{\upsilon_{k,j,l,s}}{(s+1)^{\frac{m-r}{c}}\lambda^{4-r}}\Gamma\left(\frac{4-r}{c}+1\right)}{\left(\sum_{k=0}^{\infty}\sum_{r=0}^{2}\binom{2}{r}(-1)^r\mu^r\frac{\upsilon_{k,j,l,s}}{(j+1)^{\frac{2-r}{c}}\lambda^{2-r}}\Gamma\left(\frac{2-r}{c}+1\right)\right)^{2}}.$$

6.7 Maximum likelihood

Calculating the estimation by the method of maximum likelihood estimator (MLE) for parameters a, b, c and λ, where $\mathbf{x} = \{x_1, \ldots, x_n\}^\top$ of independent random variables of size n

from the **Sec − KumW**, the log-likelihood and the score function are, respectively, presented below:

$$\ell(\theta) = n\log\left(abc\lambda^{c}\right) - \lambda^{c}\sum_{i=1}^{n} x^{c} + (c-1)\sum_{i=1}^{n}\log(x) + (a-1)\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-(\lambda\,x)^{c}}\right)$$
$$+ (b-1)\sum_{i=1}^{n}\log\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right) + \sum_{i=1}^{n}\log\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)$$
$$+ \sum_{i=1}^{n}\log\left(\sec\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right) + \sum_{i=1}^{n}\log\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right),$$

$$U_a = \frac{n}{a} + \sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-(\lambda\,x)^{c}}\right) - \frac{(b-1)\sum_{i=1}^{n}\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}} - \frac{(-1)\sum_{i=1}^{n}\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}}$$
$$-\frac{1}{3}\,\frac{\sum_{i=1}^{n}\tan\left(\frac{1}{3}\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}}$$
$$-\frac{1}{3}\,\frac{\sum_{i=1}^{n}\left(1+\left(\tan\left(\frac{1}{3}\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\tan\left(\frac{1}{3}\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)},$$

$$U_b = \frac{n}{b} + \sum_{i=1}^{n}\log\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right),$$

$$U_c = \frac{n\left(ab\lambda^{c}+abc\lambda^{c}\log(\lambda)\right)}{abc\lambda^{c}} - \lambda^{c}\log(\lambda)\,nx^{c} - \lambda^{c}nx^{c}\log(x) + n\log(x) + \frac{(a-1)\,n\,(\lambda\,x)^{c}\log(\lambda\,x)\,\mathrm{e}^{-(\lambda\,x)^{c}}}{1-\mathrm{e}^{-(\lambda\,x)^{c}}}$$
$$+\frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\log(\lambda\,x)}{1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}} + \frac{n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\log(\lambda\,x)}{1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}}$$
$$+1/3\,\frac{n\tan\left(\frac{1}{3}\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\log(\lambda\,x)}{1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}}$$
$$+\frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\log(\lambda\,x)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\tan\left(\frac{1}{3}\pi\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)},$$

and

$$
\begin{aligned}
U_\lambda = & \frac{nc}{\lambda} - \frac{\lambda^c cnx^c}{\lambda} + \frac{(a-1)\, n\, (\lambda\, x)^c\, c\mathrm{e}^{-(\lambda\, x)^c}}{\lambda\, \left(1-\mathrm{e}^{-(\lambda\, x)^c}\right)} + \frac{(b-1)\, n\, \left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a a\, (\lambda\, x)^c\, c}{\lambda\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right)} \\
& + \frac{n\, \left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a a\, (\lambda\, x)^c\, c}{\lambda\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right)} \\
& + \frac{1}{3}\, \frac{n \tan\left(1/3\, \pi\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right)\right) \pi\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right) \left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a a\, (\lambda\, x)^c\, c}{\lambda\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right)} \\
& + \frac{1}{3}\, \frac{n \left(1+\left(\tan\left(\frac{1}{3}\, \pi\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right)\right)\right)^2\right) \pi\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right) \left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a a\, (\lambda\, x)^c\, c}{\lambda\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right) \tan\left(\frac{1}{3}\, \pi\, \left(1-\left(\mathrm{e}^{-(\lambda\, x)^c}\right)^a\right)\right)}.
\end{aligned}
$$

6.8 Application

Here we use the **Sec-KumW** distribution in an application to a real data set. The data represent the lifetimes of 50 devices by Aarset [1].

Table 6.2: Lifetimes of 50 devices

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Table 6.3: Descriptive statistics.

Min.	Q_1	Median	Mean	Q_3	Max.	Var.
0.10	13.50	48.50	45.69	81.25	86.00	1078.153

We can see that the new distribution, when compared to the others, proved to get better statistics, according to the tests. Thus, we conclude that this distribution is quite flexible in the modeling of the proposed data. We can observe that the Figure 6.3 an excellent fit to the data distribution to the adequacy of the data.

Table 6.4: MLE of the parameters of the Sec-KumW, KumWP, Kum-W and W models with error in parentheses and AIC, BIC, CAIC and HQIC statistics

Distributions	Estimates					AIC	BIC	CAIC	HQIC	A^*	W^*
$SecKumW(a,b,c,\lambda)$	0.01	0.39	5.61	0.02	(-)	450.18	451.06	457.82	453.09	0.16	1.24
	(0.01)	(0.07)	(0.05)	(0.00)	(-)						
$KumWP(\lambda,a,b,c,\beta)$	0.12	0.24	3.41	0.90	0.02	462.30	471.86	463.66	465.94	1.47	0.21
	(0.37)	(0.29)	(10.73)	(7.22)	(0.04)						
$KumW(a,b,c,\beta)$	0.09	0.06	1.61	0.10	(-)	475.94	476.83	483.59	478.85	0.33	2.16
	(0.00)	(0.01)	(0.00)	(0.00)	(-)						
$W(\alpha,\beta)$	0.95	0.02	(-)	(-)	(-)	486.00	486.26	489.82	487.45	0.50	3.01
	(0.12)	(0.00)	(-)	(-)	(-)						

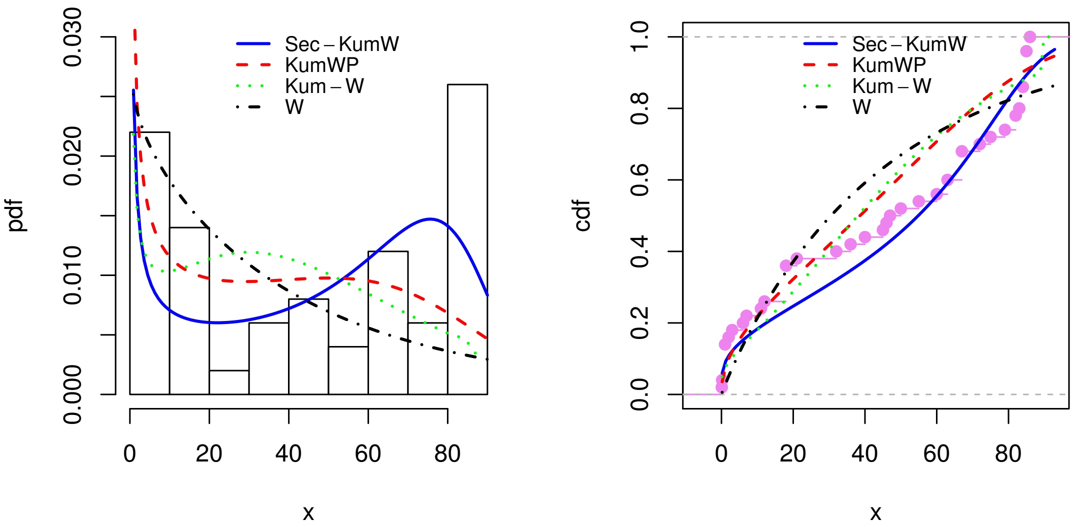


Figure 6.3: Estimated fitted density and cumulative functions for Aarset, (solid lines)

6.9 Concluding remarks

We developed a new class, the **Sec-G** of trigonometric distributions, and, within this class, a new trigonometric distribution, the **Sec-KumW**, has been thoroughly investigated. Here, we obtained the density function, the cumulative function and its expansions. Also estimates were checked via the maximum likelihood estimation. We applied the new distribution to a real data set, which brought evidence that this model can help in the analysis of survival data and we conjecture thar it can be applied to other areas of knowledge as well.

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CHAPTER 7

Numerical Results

In this section, we conducted a numerical experiment to observe the performance of four new distributions, one in each new class, taking as baseline the continuous exponential distribution. We chose the exponential distribution for it's simplicity - it's absolutely continuous for $t > 0$, it has a single parameter $\lambda > 0$ which coincides with it's expected value, and it's variance is simply λ^2.

We use the method of Monte Carlo to estimate the λ parameter of the distribution, as well as the bias, variance, the mean square error and the error. The simulation was performed using a program written in the R programming language [1] and we propose samples of sizes n= 50, 100, 200 and 1000. For each sample, 5 thousands replicas of Monte Carlo were performed. In each replication estimated parameter the λ Adequacy Model package and the Kolmogorov statistic test were used to obtain the error between the estimated and the empirical distribution.

On the following tables, the bias, EQM and Kolmogorov-Sminornov Statistic (the maximum difference between the real and estimated cumulative distribution, denoted by ERROR) should decrease with sample size.

We noted that the statistics behave as expected. The means and variances are lower then the original λ and λ^2 of the exponential distribution, and that the estimates have a consistent positive bias increasing with λ, indicating that Maximum Likelihood underestimates λ on finite samples. The EQM of the distribution Sec-Exp had lower values, which is the measure that increases the sample size.

Table 7.1: Mean, variance, bias, mean squared errors and error for **Sin-Exp** distribution

Sample	λ	Mean	Variance	Bias	EQM	ERROR
50	0.5	0.510253	0.237113	0.010253	0.237218	0.108552
	1.0	0.939595	0.796271	0.060405	0.799920	0.117992
	1.5	1.250041	1.396479	0.249959	1.458958	0.171760
	2.0	1.532249	2.089266	0.467751	2.308057	0.222940
100	0.5	0.505131	0.230111	0.005131	0.230137	0.071086
	1.0	0.931751	0.777596	0.068249	0.782253	0.085588
	1.5	1.243341	1.375575	0.256659	1.441449	0.147910
	2.0	1.523412	2.058408	0.476588	2.285544	0.201588
200	0.5	0.502832	0.227209	0.002832	0.227217	0.047356
	1.0	0.929780	0.770923	0.070220	0.775854	0.065434
	1.5	1.241337	1.366756	0.258663	1.433662	0.132035
	2.0	1.522212	2.046137	0.477788	2.274419	0.184668
1000	0.5	0.500130	0.223773	0.000130	0.223773	0.017960
	1.0	0.925736	0.762197	0.074264	0.767712	0.046451
	1.5	1.237901	1.354457	0.262099	1.423153	0.112773
	2.0	1.518615	2.030188	0.481385	2.261919	0.166131

Table 7.2: Mean, variance, bias, mean squared errors and error for **Cos-Exp** distribution

Sample	λ	Mean	Variance	Bias	EQM	ERROR
50	0.5	0.505895	0.160421	0.005895	0.160455	0.110332
	1.0	1.012201	0.639923	0.012201	0.640072	0.112180
	1.5	1.517353	1.439227	0.017353	1.439529	0.110568
	2.0	1.993526	2.488482	0.006474	2.488524	0.111420
100	0.5	0.502358	0.156810	0.002358	0.156815	0.072852
	1.0	1.005315	0.626725	0.005315	0.626754	0.072974
	1.5	1.508185	1.417891	0.008185	1.417958	0.073404
	2.0	1.982731	2.448959	0.017269	2.449258	0.073988
200	0.5	0.500843	0.155120	0.000843	0.155121	0.048272
	1.0	1.002988	0.622112	0.002988	0.622121	0.049215
	1.5	1.503227	1.396548	0.003227	1.396558	0.047629
	2.0	1.979187	2.443232	0.020813	2.443665	0.050086
1000	0.5	0.500052	0.154536	0.000052	0.154536	0.018462
	1.0	1.000401	0.615162	0.000401	0.615162	0.018435
	1.5	1.500524	1.388212	0.000524	1.388213	0.018598
	2.0	1.974706	2.414460	0.025294	2.415100	0.020491

Table 7.3: Mean, variance, bias, mean squared errors and error for **Tan-Exp** distribution

Sample	λ	Mean	Variance	Bias	EQM	ERROR
50	0.5	0.511776	0.235926	0.011776	0.236065	0.107144
	1.0	1.022400	0.935127	0.022400	0.935629	0.105608
	1.5	1.514971	2.057692	0.014971	2.057916	0.107168
	2.0	1.928897	3.311094	0.071103	3.316149	0.111560
100	0.5	0.504533	0.225463	0.004533	0.225483	0.069846
	1.0	1.008557	0.899364	0.008557	0.899437	0.069782
	1.5	1.502189	2.000083	0.002189	2.000087	0.070118
	2.0	1.912306	3.228828	0.087694	3.236518	0.077196
200	0.5	0.502743	0.223742	0.002743	0.223750	0.046143
	1.0	1.003380	0.890281	0.003380	0.890292	0.046813
	1.5	1.494661	1.982587	0.005339	1.982616	0.046299
	2.0	1.903076	3.201971	0.096924	3.211366	0.055622
1000	0.5	0.500401	0.220628	0.000401	0.220628	0.017550
	1.0	1.000876	0.880072	0.000876	0.880072	0.017682
	1.5	1.489265	1.955509	0.010735	1.955624	0.018021
	2.0	1.896916	3.171492	0.103084	3.182119	0.034165

Table 7.4: Mean, variance, bias, mean squared errors and error for **Sec-Exp** distribution

Sample	λ	Mean	Variance	Bias	EQM	ERROR
50	0.5	0.505315	0.115616	0.005315	0.115645	0.110072
	1.0	1.008779	0.462582	0.008779	0.462659	0.110636
	1.5	1.513826	1.041856	0.013826	1.042047	0.109396
	2.0	2.019047	1.853504	0.019047	1.853867	0.110200
100	0.5	0.501622	0.114079	0.001622	0.114081	0.073808
	1.0	1.005145	0.457802	0.005145	0.457829	0.073472
	1.5	1.506766	1.030711	0.006766	1.030757	0.073868
	2.0	2.006363	1.824579	0.006363	1.824619	0.073030
200	0.5	0.501281	0.114186	0.001281	0.114188	0.047979
	1.0	1.002277	0.456581	0.002277	0.456586	0.047851
	1.5	1.502043	1.022065	0.002043	1.022069	0.049010
	2.0	2.006762	1.827905	0.006762	1.827951	0.048116
1000	0.5	0.500164	0.112767	0.000164	0.112767	0.018422
	1.0	1.000108	0.452988	0.000108	0.452988	0.018222
	1.5	1.500893	1.017272	0.000893	1.017273	0.018410
	2.0	2.001038	1.819812	0.001038	1.819813	0.018346

Bibliography

[1] R Development Core Team, R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, Vienna, 2012.

CHAPTER 8

Conclusions, contributions and future work

8.1 Conclusions

In this thesis we proposed new classes of probabilistic trigonometric distributions and some distributions within each class. We use some continuous functions, such as Reverse Weibull, Weibull, Burr XII and Kumaraswamy distributions as baselines.

The initial objective was the proposal of only one trigonometric class, the exponential sine. But with a larger number of distributions, we understand that greater diversity would produce greater range and a huge number of applicability. Thus, these goals have been achieved satisfactorily since we obtained the properties for each class, as well as good fitting for each of the new models proposed.

The four presented distributions have been successful in their applications and have become good proposals in addressing each problem. Furthermore, we report that there were some issues regarding the parameter estimates and an implementation of AdequacyModel package was made only to facilitate the opening kickoff values by observing that there was a sensitivity in the kicks, creating impact in the statistics test such as Anderson-Darling for example.

8.2 Contributions

This thesis contributes to the Statistical Science with some classes and models. The first class, the Sin-G class, generates an infinite number of distributions, but we present the **Sin-IW** as a special case.

The second contribution comes from the Cos-G class, which also has an infinite number of distributions, but we present the **Cos-W** as a special case.

The third contribution comes from the Tan-G class, which, just like the two aforementioned ones, has an infinite number of distributions, but we suggest, as a special case, the **Tan-BXII**.

The fourth contribution comes from the Sec-G class, which, just like the other ones above, has an infinite number of distributions, but we propose the **Sec-KumW** as a special case.

We understand that each distribution contributed to the proposed objectives: the creation of new distributions with a varied number of applications.

8.3 Future work

We are working on building a single class involving all trigonometric classes proposed in this thesis, as well as applications which are more frequently used such as Beta, Gamma and Maxwell. This is the focus of my future work, as well as the use of discrete distributions as the baseline and the construction of new classes containing the inverse trigonometric and hyperbolic classes. I have recently developed a new class which I have named: $\mathbf{Exp_Sin - G}$, to be used as the baseline of the proposed trigonometric class in this thesis as well as the exponential distribution. As for the G it is initially the Weibull distribution as it can be seen below:

$$H_G = \int_0^{\frac{sin[\frac{\pi}{2}G(x)]}{1-sin[\frac{\pi}{2}G(x)]}} \lambda \exp(-\lambda t) dt \tag{8.1}$$

This new distribution,from 46 observations, was applied in data contained in Article "The beta Weibull Poisson distribution", which concern the active repair time maintenance of airborne communicational transceiver. The new distribution with five parameters was generated and the application held; however, we observed that the new $\mathbf{Exp_Sin - W}$ distribution proved to be much more flexible with only three parameters.

CHAPTER 9

Appendix A

In this appendix we will develop the partial derivatives of maximum likelihood.

9.1 Sin-IW

Related to the parameters α and θ and observed information matrix

$$J(\theta) = -\begin{pmatrix} U_{\alpha\alpha} & U_{\alpha\theta} \\ . & U_{\theta\theta} \end{pmatrix} \tag{9.1}$$

$$
\begin{aligned}
U_{\alpha\theta} = {} & \frac{\alpha\pi^2}{4}\sum_{i=1}^{n}\left(\exp(-\alpha x^{-\theta})\right)^2 x^{-2\theta}\log(x) + \frac{\alpha\pi}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right) \\
& \times \exp(-\alpha x^{-\theta})x^{-2\theta}\log(x) - \frac{\pi}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right)x^{-\theta}\log(x) \\
& + \frac{\alpha\pi^2}{4}\sum_{i=1}^{n}\left(\exp(-\alpha x^{-\theta})\right)^2 x^{-2\theta}\log(x) + \frac{\alpha\pi^2}{4}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right)^2 \\
& \times x^{-2\theta}\log(x).
\end{aligned} \tag{9.2}
$$

$$
\begin{aligned}
U_{\alpha\alpha} = {} & -\frac{n}{\alpha} - \frac{\pi}{4}\sum_{i=1}^{n}\left(x_i^{-\theta}\right)\left(\exp(-\alpha x_i^{-\theta})\right) - \frac{\pi}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right) \\
& \times \exp(-\alpha x_i^{-\theta})\left(x_i^{-\theta}\right) - \frac{\pi}{4}\sum_{i=1}^{n}\left(x_i^{-\theta}\right)\tan\left(\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right)\left(\exp(-\alpha x_i^{-\theta})\right).
\end{aligned} \tag{9.3}
$$

$$
\begin{aligned}
U_{\theta\theta} = {} & -\frac{n}{\theta^2} - \frac{\alpha^2\pi^2}{4}\sum_{i=1}^{n}\left(\exp(-\alpha x_i^{-\theta})\right)^2\left(x_i^{-\theta}\right)^2\ln(x_i^2) - \frac{\alpha^2\pi}{2}\sum_{i=1}^{n}\left(x_i^{-\theta}\right)^2\ln(x)^2 \\
& \times \tan\left(\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right)\exp(-\alpha x_i^{-\theta}) + \frac{\alpha\pi}{2}\sum_{i=1}^{n}\left(x_i^{-\theta}\right)\ln x_i^2\tan\left(\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right) \\
& \times \exp(-\alpha x_i^{-\theta}) - \frac{(\pi\alpha)^2}{4}\sum_{i=1}^{n}\left(x_i^{-\theta}\right)^2\ln x_i^2\tan\left(\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right)^2\left(\exp(-\alpha x_i^{-\theta})\right)^2.
\end{aligned}
$$

9.2 Cos-W

Related to the parameters α and λ and observed information matrix

$$
J(\theta) = -\begin{pmatrix} U_{\alpha\alpha} & U_{\alpha\lambda} \\ . & U_{\lambda\lambda} \end{pmatrix} \tag{9.4}
$$

$$
\begin{aligned}
U_{\alpha\alpha} = & -\frac{n\left(2\lambda^{\alpha}\ln(\lambda)+\alpha\lambda^{\alpha}\ln(\lambda)^{2}\right)}{\alpha\lambda^{\alpha}} - \frac{n\left(\lambda^{\alpha}+\alpha\lambda^{\alpha}\ln(\lambda)\right)}{\alpha^{2}\lambda^{\alpha}} - \frac{n\left(\lambda^{\alpha}+\alpha\lambda^{\alpha}\ln(\lambda)\right)\ln(\lambda)}{\alpha\lambda^{\alpha}} \\
& - \frac{\pi^{2}}{4}\sum_{i=1}^{n}\ln(\lambda x_i)^{2}\times\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right)^{2} \qquad (9.5) \\
& - \frac{\pi}{2}\sum_{i=1}^{n}\ln(\lambda x_i)^{2}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right)\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right)^{2} \\
& - \frac{\pi}{2}\sum_{i=1}^{n}\ln(\lambda x_i)^{2}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right)\left((\lambda x_i)^{\alpha}\right)\left(\exp(-(\lambda x_i)^{\alpha})\right) \\
& + \frac{\pi}{2}\sum_{i=1}^{n}\ln(\lambda x_i)^{2}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right)(1-exp(-(\lambda x_i)^{\alpha}))^{\beta-1}\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right) \\
& + \frac{\pi}{2}\sum_{i=1}^{n}\ln(\lambda x_i)^{2}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right)\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right)^{2} \\
& + \frac{\pi^{2}}{4}\sum_{i=1}^{n}\ln(\lambda x_i)^{2}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right)^{2}(1-exp(-(\lambda x_i)^{\alpha}))^{2\beta-2}\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right)^{2}. \qquad (9.6)
\end{aligned}
$$

$$
\begin{aligned}
U_{\lambda\lambda} = & -\frac{n\alpha}{\lambda^{2}} - \frac{\alpha^{2}\pi^{2}}{4\lambda^{2}}\sum_{i=1}^{n}\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right)^{2} \\
& - \frac{\alpha^{2}\beta^{2}\pi}{2\lambda^{2}}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right)\left((\lambda x)^{\alpha}\right)^{2} \\
& \times\left(\exp(-(\lambda x)^{\alpha})\right)^{2} - \frac{\alpha^{2}\beta\pi}{2\lambda^{2}}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right) \\
& \times\left((\lambda x)^{\alpha}\right)\left(\exp(-(\lambda x_i)^{\alpha})\right) + \frac{\alpha\pi}{2\lambda^{2}}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right) \\
& \times\left((\lambda x_i)^{\alpha}\right)\left(\exp(-(\lambda x_i)^{\alpha})\right) + \frac{\alpha^{2}\pi}{2\lambda^{2}}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))^{\beta}\right) \\
& \times\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right) + \frac{\alpha^{2}\pi}{2\lambda^{2}}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))^{\beta}\right) \\
& \times\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right)^{2} - \frac{\alpha^{2}\pi^{2}}{4\lambda^{2}}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-exp(-(\lambda x_i)^{\alpha}))\right)^{2} \\
& \times\left((\lambda x_i)^{\alpha}\right)^{2}\left(\exp(-(\lambda x_i)^{\alpha})\right)^{2}. \qquad (9.7)
\end{aligned}
$$

$$
\begin{aligned}
U_{\alpha\lambda} = {} & \frac{n}{\lambda} - \frac{\alpha\pi^2\lambda^{2\alpha-1}}{4}\sum_{i=1}^{n}(1-\exp(-(\lambda x)^{\alpha}))x^{2\alpha}\log(\lambda x)\exp(-(\lambda x)^{2\alpha}) \\
& - \frac{\alpha\pi\lambda^{2\alpha-1}}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-\exp(-(\lambda x)^{\alpha}))\right)x^{2\alpha}\log(\lambda x)\exp(-(\lambda x)^{2\alpha}) \\
& - \frac{\alpha\pi\lambda^{2\alpha-1}}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-\exp(-(\lambda x)^{\alpha}))\right)x^{\alpha}\log(\lambda x)\exp(-(\lambda x)^{\alpha}) \\
& - \frac{\alpha\pi\lambda^{\alpha-1}}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-\exp(-(\lambda x)^{\alpha}))\right)x^{2\alpha}\exp(-(\lambda x)^{\alpha}) \\
& + \frac{\alpha\pi\lambda^{2\alpha-1}}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-\exp(-(\lambda x)^{\alpha}))\right)x^{2\alpha}\log(\lambda x)\exp(-(\lambda x)^{\alpha}) \\
& + \frac{\alpha\pi\lambda^{2\alpha-1}}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-\exp(-(\lambda x)^{\alpha}))\right)x^{2\alpha}\exp(-(\lambda x)^{2\alpha})\log(\lambda x) \\
& - \frac{\alpha\pi^2\lambda^{2\alpha-1}}{4}\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}(1-\exp(-(\lambda x)^{\alpha}))\right)^2 x^{2\alpha}\exp(-(\lambda x)^{2\alpha})\log(\lambda x).
\end{aligned} \tag{9.8}
$$

9.3 Tan-BurrXII

Related to the parameters c, k and s and observed information matrix

$$
J(\theta) = -\begin{pmatrix} U_{cc} & U_{ck} & U_{cs} \\ . & U_{kk} & U_{ks} \\ . & . & U_{ss} \end{pmatrix} \tag{9.9}
$$

$$
\begin{aligned}
U_{ck} &= \sum_{i=1}^{n} \frac{\left(\frac{x_i}{s}\right)^c \ln\left(\frac{x_i}{s}\right)}{1+\left(\frac{x_i}{s}\right)^c} \\
&+ \sum_{i=1}^{n} \log\left\{ \frac{\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)(-k-1)\left(\frac{x_i}{s}\right)^c \log\left(\left(\frac{x_i}{s}\right)\right)\log\left(1+\left(\frac{x_i}{s}\right)^c\right)}{1+\left(\frac{x_i}{s}\right)^c} \right\} \\
&+ \sum_{i=1}^{n} \log\left\{ \frac{\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^c \log\left(\left(\frac{x_i}{s}\right)\right)}{1+\left(\frac{x_i}{s}\right)^c} \right\} \\
&- \frac{(-k-1)\pi^2}{8} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-2(k+1)}\right)\left(\frac{x_i}{s}\right)^c \log\left(\frac{x_i}{s}\right)\log\left(1+\left(\frac{x_i}{s}\right)^c\right)}{1+\left(\frac{x_i}{s}\right)^c} \\
&- \frac{(k+1)\pi}{2} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^c \ln\left(\frac{x_i}{s}\right)\log\left(1+\left(\frac{x_i}{s}\right)^c\right)}{1+\left(\frac{x_i}{s}\right)^c} \\
&- \frac{(k+1)\pi}{2} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^c \log\left(\frac{x_i}{s}\right)}{1+\left(\frac{x_i}{s}\right)^c}. \qquad (9.10)
\end{aligned}
$$

$$
\begin{aligned}
U_{cs} &= -\frac{n}{s} + \frac{(k+1)\sum_{i=1}^{n}\left(\frac{x_i}{s}\right)^c \log\left(\frac{x_i}{s}\right) c + (k+1)\sum_{i=1}^{n}\left(\frac{x_i}{s}\right)^c}{s\left(1+\left(\frac{x_i}{s}\right)^c\right)} \\
&- \frac{(k+1)\sum_{i=1}^{n}\left(\frac{x_i}{s}\right)^{2c}\log\left(\frac{x_i}{s}\right) c}{s\left(1+\left(\frac{x_i}{s}\right)^c\right)} \\
&- \frac{(-k-1)^2\pi^2}{8} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^c \log\left(\frac{x_i}{s}\right)\log\left(1+\left(\frac{x_i}{s}\right)^c\right)}{1+\left(\frac{x_i}{s}\right)^c} \\
&- \frac{(-k-1)^2\pi}{2} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^{2c} \log\left(\frac{x_i}{s}\right)}{\left(1+\left(\frac{x_i}{s}\right)^c\right)^2 s}. \qquad (9.11)
\end{aligned}
$$

$$U_{ks} = \frac{\sum_{i=1}^{n} \left(\frac{x_i}{s}\right)^c c}{\left(1+\left(\frac{x_i}{s}\right)^c\right) s}$$

$$(-1)\sum_{i=1}^{n} \log \left\{ \frac{\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right) \log\left(1+\left(\frac{x_i}{s}\right)^c\right)(-k-1)\left(\frac{x_i}{s}\right)^c c}{\left(1+\left(\frac{x_i}{s}\right)^c\right)} \right\}$$

$$\frac{(-k-1)\pi^2}{8} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^{2c} \log\left(\frac{x_i}{s}\right)}{\left(1+\left(\frac{x_i}{s}\right)^c\right) s}$$

$$\frac{(-k-1)\pi}{2} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^{c} \log\left(\frac{x_i}{s}\right) \log\left(1+\left(\frac{x_i}{s}\right)^c\right)}{\left(1+\left(\frac{x_i}{s}\right)^c\right) s}$$

$$\frac{c\pi}{2} \sum_{i=1}^{n} \frac{\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}\right)\left(\frac{x_i}{s}\right)^{c} \left(1+\left(\frac{x_i}{s}\right)^c\right)^{-k-1}}{\left(1+\left(\frac{x_i}{s}\right)^c\right) s}. \tag{9.12}$$

$$U_{cc} = \frac{n\left(-2\,ks^{-c}\log(s) + cks^{-c}(\log(s))^2\right)}{cks^{-c}} - \frac{n\left(ks^{-c} - cks^{-c}\log(s)\right)}{c^2ks^{-c}} + \frac{n\left(ks^{-c} - cks^{-c}\log(s)\right)\log(s)}{cks^{-c}}$$
$$-(k+1)\sum_{i=1}^{n}\left(\frac{x}{s}\right)^c\left(\log\left(\frac{x}{s}\right)\right)^2\left(1+\left(\frac{x}{s}\right)^c\right)^{-1} + (k+1)\sum_{i=1}^{n}\left(\left(\frac{x}{s}\right)^c\right)^2\left(\log\left(\frac{x}{s}\right)\right)^2\left(1+\left(\frac{x}{s}\right)^c\right)^{-2}$$
$$+\sum_{i=1}^{n}(ln)\left\{-(-k-1)^2\left(1+\left(\frac{x}{s}\right)^c\right)^{-k-1}\left(\left(\frac{x}{s}\right)^c\right)^2\left(\log\left(\frac{x}{s}\right)\right)^2\left(1+\left(\frac{x}{s}\right)^c\right)^{-2}\right\}$$
$$\sum_{i=1}^{n}\log\left\{+(-k-1)\left(1+\left(\frac{x}{s}\right)^c\right)^{-k-1}\left(\left(\frac{x}{s}\right)^c\right)^2\left(\log\left(\frac{x}{s}\right)\right)^2\left(1+\left(\frac{x}{s}\right)^c\right)^{-2}\right\}. \tag{9.13}$$

$$U_{kk} = -\frac{n}{k^2} + (\beta-1)\sum_{i=1}^{n}(ln)\left\{-\left(1+\left(\frac{x}{s}\right)^c\right)^{-k-1}\left(\ln\left(1+\left(\frac{x}{s}\right)^c\right)\right)^2\right\}$$
$$+1/8\sum_{i=1}^{n}\left(1+\left(\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x}{s}\right)^c\right)^{-k-1}\right)\right)^2\right)\pi^2\left(\left(1+\left(\frac{x}{s}\right)^c\right)^{-k-1}\right)^2$$
$$\times\left(\ln\left(1+\left(\frac{x}{s}\right)^c\right)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\tan\left(\frac{\pi}{4}\left(1+\left(\frac{x}{s}\right)^c\right)^{-k-1}\right)\pi\left(1+\left(\frac{x}{s}\right)^c\right)^{-k-1}$$
$$\times\left(\ln\left(1+\left(\frac{x}{s}\right)^c\right)\right)^2. \tag{9.14}$$

$$
\begin{aligned}
U_{ss} &= \frac{nc}{s^2} - \frac{(k+1)\sum_{i=1}^{n} c^2}{s^2}\left(\frac{x}{s}\right)^c \left(1+\left(\frac{x}{s}\right)^c\right)^{-1} \\
&- \frac{(k+1)\,c}{s^2}\sum_{i=1}^{n}\left(\frac{x}{s}\right)^c \left(1+\left(\frac{x}{s}\right)^c\right)^{-1} + \frac{(k+1)\,nc^2}{s^2}\left(\left(\frac{x}{s}\right)^c\right)^2 \left(1+\left(\frac{x}{s}\right)^c\right)^{-2}.
\end{aligned}
$$

9.4 Sec-KumW

In this appendix we will develop the partial derivatives of maximum likelihood regarding the parameters a, b, c and λ and observed information matrix

$$
J(\theta) = -\begin{pmatrix} U_{aa} & U_{ab} & U_{ac} & U_{a\lambda} \\ . & U_{bb} & U_{bc} & U_{b\lambda} \\ . & . & U_{cc} & U_{c\lambda} \\ . & . & . & U_{\lambda\lambda} \end{pmatrix} \qquad (9.15)
$$

$$U_{aa} = -\frac{n}{a^2} - \frac{(b-1)\,n\left(e^{-(\lambda x)^c}\right)^a \left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{1-\left(e^{-(\lambda x)^c}\right)^a} - \frac{(b-1)\,n\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2 \left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2} \tag{9.16}$$

$$- \frac{(\beta-1)\,n\left(e^{-(\lambda x)^c}\right)^a \left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{1-\left(e^{-(\lambda x)^c}\right)^a} - \frac{(\beta-1)\,n\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2 \left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2}$$

$$+ \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\right)^2\right)\pi^2\left(\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)^2\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2}$$

$$+ \frac{1}{3}\,\frac{n\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2}$$

$$- \frac{1}{3}\,\frac{n\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\left(e^{-(\lambda x)^c}\right)^a\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{1-\left(e^{-(\lambda x)^c}\right)^a}$$

$$- \frac{1}{3}\,\frac{n\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2}$$

$$+ \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\right)^2\right)\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)}$$

$$- \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\right)^2\right)\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\left(e^{-(\lambda x)^c}\right)^a\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)}$$

$$- \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\right)^2\right)\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)}$$

$$- 1/9\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\right)^2\right)^2\pi^2\left(\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)^2\left(\left(e^{-(\lambda x)^c}\right)^a\right)^2\left(\ln\left(e^{-(\lambda x)^c}\right)\right)^2}{\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)^2\left(\tan\left(\frac{1}{3}\pi\left(1-\left(e^{-(\lambda x)^c}\right)^a\right)\right)\right)^2}.$$

$$U_{bb} = -\frac{n}{b^2}. \tag{9.17}$$

$$U_{ab} = -\frac{n\left(e^{-(\lambda x)^c}\right)^a \ln\left(e^{-(\lambda x)^c}\right)}{1-\left(e^{-(\lambda x)^c}\right)^a}. \tag{9.18}$$

$$U_{cb} = \frac{n\left(e^{-(\lambda x)^c}\right)^a a\,(\lambda x)^c \ln(\lambda x)}{1-\left(e^{-(\lambda x)^c}\right)^a} \tag{9.19}$$

$$U_{b\lambda} = \frac{n\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)}. \tag{9.20}$$

$$\begin{aligned}
U_{ac} &= \frac{n\,(\lambda\,x)^c\log(\lambda\,x)\,\mathrm{e}^{-(\lambda\,x)^c}}{1-\mathrm{e}^{-(\lambda\,x)^c}} + \frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\log(\lambda\,x)\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a} \\
&+ \frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a(\lambda\,x)^c\log(\lambda\,x)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a} + \frac{(b-1)\,n\left(\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)a\,(\lambda\,x)^c\log(\lambda\,x)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2} \\
&+ \frac{n\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\log(\lambda\,x)\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a} + \frac{n\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a(\lambda\,x)^c\log(\lambda\,x)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a} \\
&+ \frac{n\left(\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)a\,(\lambda\,x)^c\log(\lambda\,x)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2} \\
&- \frac{1}{3}\,n\left(1+\left(\tan\left(\frac{1}{3}\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi^2\left(\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right) \\
&\times\left(\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2 a\,(\lambda\,x)^c\log(\lambda\,x)\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right) \\
&- \frac{1}{3}\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2 a\,(\lambda\,x)^c\log(\lambda\,x)\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2} \\
&+ 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\log(\lambda\,x)\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a} \\
&+ 1/3\,\frac{n\tan\left(\frac{1}{3}\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^\beta\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a(\lambda\,x)^c\log(\lambda\,x)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a} \\
&+ 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)a\,(\lambda\,x)^c\log(\lambda\,x)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2} \\
&- \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^\beta\right)} \\
&+ \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)} \\
&+ \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(\frac{1}{3}\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a(\lambda\,x)^c\log(\lambda\,x)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\tan\left(\frac{1}{3}\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)} \\
&+ \frac{1}{3}\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)} \\
&+ \frac{1}{9}\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)^2\pi^2\left(\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)^2\left(\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^2\left(\tan\left(\frac{1}{3}\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2}.
\end{aligned} \tag{9.21}$$

$$
\begin{aligned}
U_{a\lambda} &= \frac{n\,(\lambda\,x)^{c}\,c\mathrm{e}^{-(\lambda\,x)^{c}}}{\lambda\,\left(1-\mathrm{e}^{-(\lambda\,x)^{c}}\right)} + \frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a} a\,(\lambda\,x)^{c}\,c\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)} + \frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}(\lambda\,x)^{c}\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)} \\
&+ \frac{(b-1)\,n\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)a\,(\lambda\,x)^{c}\,c}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\lambda} + \frac{n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a} a\,(\lambda\,x)^{c}\,c\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)} \\
&+ \frac{n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}(\lambda\,x)^{c}\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)} + \frac{(\beta-1)\,n\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)a\,(\lambda\,x)^{c}\,c}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\lambda} \\
&- 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi^{2}\left(\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)^{2}\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\lambda} \\
&- 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2} a\,(\lambda\,x)^{c}\,c\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\lambda} \\
&+ 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a} a\,(\lambda\,x)^{c}\,c\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)} \\
&+ 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{\beta}\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}(\lambda\,x)^{c}\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)} \\
&+ 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{\beta}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)a\,(\lambda\,x)^{c}\,c}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\lambda} \\
&- 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2} a\,(\lambda\,x)^{c}\,c\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\lambda\,\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)} \\
&+ 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a} a\,(\lambda\,x)^{c}\,c\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)} \\
&+ 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}(\lambda\,x)^{c}\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)} \\
&+ 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)a\,(\lambda\,x)^{c}\,c}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\lambda\,\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)} \\
&+ 1/9\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{\beta}\right)\right)^{2}\right)^{2}\pi^{2}\left(\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)^{2}\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\log\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)a\,(\lambda\,x)^{c}\,c}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\lambda}.
\end{aligned}
\tag{9.22}
$$

$$
\begin{aligned}
U_{a\beta} =& -\frac{n\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a \log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a}\\
&-1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi^2\left(\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)^2\log\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a}\\
&-1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\log\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a}\\
&-1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a}\\
&-1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\log\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)}\\
&-1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\log\left(\mathrm{e}^{-(\lambda\,x)^c}\right)}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)}\\
&+1/9\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)^\beta\right)\right)^2\right)^2\pi^2\left(\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)^2\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a}{\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2}. \qquad (9.23)
\end{aligned}
$$

$$
\begin{aligned}
U_{c\lambda} = {} & \frac{n}{\lambda} + 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a\,(\lambda x)^{c}}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)} \\
& - \frac{(b-1)\,n\left(\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2} a^{2}\left((\lambda x)^{c}\right)^{2} c\log(\lambda x)}{\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}\lambda} - \frac{n\left(\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2} a^{2}\left((\lambda x)^{c}\right)^{2} c\log(\lambda x)}{\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}\lambda} \\
& + 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi^{2}\left(\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)^{2}\left(\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2} c}{\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}\lambda} \\
& - 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a^{2}\left((\lambda x)^{c}\right)^{2}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)} \\
& + 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a\,(\lambda x)^{c}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)} \\
& - 1/9\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)^{2}\pi^{2}\left(\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)^{2}\left(\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}}{\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\right)^{2}\lambda} \\
& - 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a^{2}\left((\lambda x)^{c}\right)^{2}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)} \\
& + 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a\,(\lambda x)^{c}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)} \\
& + 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a\,(\lambda x)^{c}}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)} - \frac{\lambda^{c} n x^{c}}{\lambda} \\
& - 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2} a^{2}\left((\lambda x)^{c}\right)^{2} c\log(\lambda x)}{\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}\lambda} \\
& - 1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2} a^{2}\left((\lambda x)^{c}\right)^{2} c\log(\lambda x)}{\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}\lambda\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)} \\
& + \frac{(a-1)\,n\,(\lambda x)^{c}\,\mathrm{e}^{-(\lambda x)^{c}}}{\lambda\,\left(1-\mathrm{e}^{-(\lambda x)^{c}}\right)} - \frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a^{2}\left((\lambda x)^{c}\right)^{2}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)} + \frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a\,(\lambda x)^{c}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)} \\
& - \frac{n\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a^{2}\left((\lambda x)^{c}\right)^{2}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)} + \frac{n\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a} a\,(\lambda x)^{c}\log(\lambda x)\,c}{\lambda\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)} \\
& + 1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2} a^{2}\left((\lambda x)^{c}\right)^{2}\log(\lambda x)\,c}{\left(1-\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{a}\right)^{2}\lambda} \\
& - \frac{(a-1)\,n\left((\lambda x)^{c}\right)^{2} c\left(\mathrm{e}^{-(\lambda x)^{c}}\right)^{2}\log(\lambda x)}{\lambda\,\left(1-\mathrm{e}^{-(\lambda x)^{c}}\right)^{2}} +
\end{aligned}
\tag{9.24}
$$

$$1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(\left(e^{-(\lambda\,x)^c}\right)^a\right)^2 a^2\left((\lambda\,x)^c\right)^2\log(\lambda\,x)\,c}{\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)^2\lambda\,\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)}$$
$$-\frac{\lambda^c\log(\lambda)\,cnx^c}{\lambda}-\frac{\lambda^c cnx^c\log(x)}{\lambda}+\frac{(a-1)\,n\,(\lambda\,x)^c\log(\lambda\,x)\,ce^{-(\lambda\,x)^c}}{\lambda\,\left(1-e^{-(\lambda\,x)^c}\right)}-\frac{(a-1)\,n\left((\lambda\,x)^c\right)^2 c\log(\lambda\,x)\,e^{-(\lambda\,x)^c}}{\lambda\,\left(1-e^{-(\lambda\,x)^c}\right)}$$
$$+\frac{(b-1)\,n\left(e^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c}{\lambda\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)}+\frac{n\left(e^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c}{\lambda\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)}.$$

$$U_{\lambda\lambda}=-\frac{(b-1)\,n\left(e^{-(\lambda\,x)^c}\right)^a a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)}-\frac{n\left(e^{-(\lambda\,x)^c}\right)^a a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)}-\frac{(a-1)\,n\,(\lambda\,x)^c\,ce^{-(\lambda\,x)^c}}{\lambda^2\left(1-e^{-(\lambda\,x)^c}\right)}$$
$$-1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(e^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\,c}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)}$$
$$-1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(e^{-(\lambda\,x)^c}\right)^a a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)}$$
$$-1/9\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)^2\pi^2\left(\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)^2\left(\left(e^{-(\lambda\,x)^c}\right)^a\right)^2a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)^2\left(\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2}$$
$$-\frac{nc}{\lambda^2}-\frac{(b-1)\,n\left(\left(e^{-(\lambda\,x)^c}\right)^a\right)^2a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)^2}-\frac{n\left(\left(e^{-(\lambda\,x)^c}\right)^a\right)^2a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)^2}$$
$$+1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(e^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\,c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)}$$
$$-1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(e^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\,c}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)}+\frac{\lambda^c cnx^c}{\lambda^2}$$
$$-1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(\left(e^{-(\lambda\,x)^c}\right)^a\right)^2a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)^2}$$
$$+1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(\left(e^{-(\lambda\,x)^c}\right)^a\right)^2a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)^2}$$
$$+1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(e^{-(\lambda\,x)^c}\right)^a a\,(\lambda\,x)^c\,c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)}$$
$$+1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)\right)^2\right)\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\left(\left(e^{-(\lambda\,x)^c}\right)^a\right)^2a^2\left((\lambda\,x)^c\right)^2c^2}{\lambda^2\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)^2\tan\left(1/3\,\pi\,\left(1-\left(e^{-(\lambda\,x)^c}\right)^a\right)\right)}$$
$$-\frac{\lambda^c c^2nx^c}{\lambda^2}-\frac{(a-1)\,n\left((\lambda\,x)^c\right)^2c^2e^{-(\lambda\,x)^c}}{\lambda^2\left(1-e^{-(\lambda\,x)^c}\right)} \tag{9.25}$$

$$
\begin{aligned}
&-1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}a^{2}\left((\lambda\,x)^{c}\right)^{2}c^{2}}{\lambda^{2}\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)}\\
&+\frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\,c^{2}}{\lambda^{2}\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)}+\frac{n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\,c^{2}}{\lambda^{2}\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)}-\frac{(a-1)\,n\left((\lambda\,x)^{c}\right)^{2}c^{2}\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{2}}{\lambda^{2}\left(1-\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{2}}\\
&+1/3\,\frac{n\left(1+\left(\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\right)^{2}\right)\pi^{2}\left(\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)^{2}\left(\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}a^{2}\left((\lambda\,x)^{c}\right)^{2}c^{2}}{\lambda^{2}\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)^{2}}\\
&-1/3\,\frac{n\tan\left(1/3\,\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\right)\pi\,\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a^{2}\left((\lambda\,x)^{c}\right)^{2}c^{2}}{\lambda^{2}\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)}\\
&+\frac{(a-1)\,n\,(\lambda\,x)^{c}\,c^{2}\mathrm{e}^{-(\lambda\,x)^{c}}}{\lambda^{2}\left(1-\mathrm{e}^{-(\lambda\,x)^{c}}\right)}-\frac{(b-1)\,n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\,c}{\lambda^{2}\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)}-\frac{n\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}a\,(\lambda\,x)^{c}\,c}{\lambda^{2}\left(1-\left(\mathrm{e}^{-(\lambda\,x)^{c}}\right)^{a}\right)}.
\end{aligned}
$$

CHAPTER 10

Appendix B

In this chapter we present the codes used in this thesis.

```
###########################################################################
#                   UNIVERSIDADE FEDERAL RURAL DE PERNAMBUCO              #
#          PROGAMA DE POS-GRADUACAO EM BIOMETRIA E ESTATISTICA APLICADA   #
#                    CODIGO FONTE DOS GRAFICOS E ESTIMATIVAS              #
#                              BY LUCIANO SOUZA                           #
###########################################################################

#-------------------INICIO DA DISTRIBUICAO SIN-IW-------------------------

#------------------------------PDF----------------------------------------

require(pracma)
require(stats)
require(AdequacyModel)
```

```
par(pty="s")
RSinIW=function(x,alpha,theta){
((pi/2))*(alpha*theta*x^(-theta-1)*exp(-alpha*x^(-theta))*
cos((pi/2)*(exp(-alpha*x^(-theta))))/(1-sin((pi/2)*
(exp(-alpha*x^(-theta)))))
}
x = seq(0, 5, length = 1000)
alpha =1.5; theta=2.0
alpha1=1.0; theta1=1.0
alpha2=0.5; theta2=0.3
alpha3=2.0; theta3=.5
alpha4=3.5; theta4=1.5

curve(RSinIW(x,alpha,theta),cex.lab=0.9,cex.main=0.5,cex.axis=0.9,lty=1,
lwd=2,ylim=c(0,2),type="l",from=0,to=5,xlab="x",
ylab="Hazard")
G1=RSinIW(x,alpha1,theta1)
G2=RSinIW(x,alpha2,theta2)
G3=RSinIW(x,alpha3,theta3)
G4=RSinIW(x,alpha4,theta4)
lines(x,G1,lty=2,lwd=2)
lines(x,G2,lty=3,lwd=2)
lines(x,G3,lty=4,lwd=2)
lines(x,G4,lty=5,lwd=2)
legend('topright',c(expression(
list(alpha==1.5, theta==2.0),
list(alpha==1.5, theta==1.0),
list(alpha==0.5, theta==0.3),
list(alpha==2.0, theta==0.5),
list(alpha==3.5, theta==1.5))),ncol=1,bty="n",
lty=c(1,2,3,4,5),cex=0.7,lwd=2)
```

```
#-------------------------------------------------------------------------------------

par(pty="s")
SSinIW=function(x,alpha,theta){ #Sobrevivencia
1-sin((pi/2)*(exp(-alpha*x^(-theta))))
}
x = seq(0, 5, length = 1000)
alpha =0.5; theta=1.1
alpha1=1.0; theta1=1.0
alpha2=3.5; theta2=0.5
alpha3=2.1; theta3=0.9
alpha4=1.5; theta4=0.5
curve(SSinIW(x,alpha,theta),cex.lab=0.9,cex.main=0.5,
cex.axis=0.9,lty=1,lwd=2,ylim=c(0,1), type="l",
from=0,to=5,xlab="x",ylab="Survival")
G1=SSinIW(x,alpha1,theta1)
G2=SSinIW(x,alpha2,theta2)
G3=SSinIW(x,alpha3,theta3)
G4=SSinIW(x,alpha4,theta4)
lines(x,G1,lty=2, lwd=2)
lines(x,G2,lty=3, lwd=2)
lines(x,G3,lty=4, lwd=2)
lines(x,G4,lty=5, lwd=2)
legend('topright',c(expression(
list(alpha==0.5, theta==1.1),
list(alpha==1.0, theta==1.9),
list(alpha==3.5, theta==0.5),
list(alpha==2.1, theta==0.9),
list(alpha==1.5, theta==0.5))),ncol=1,bty="n"
,lty=c(1:5),cex =0.7,lwd=2)
```

```
#-----------------------------------------------------------

x=scan()
120.0 105.0  98.0  96.0  98.0  87.0 158.4  99.4 184.0 106.7
237.0 234.0  81.0  84.0 109.0 239.0 360.0  84.0  77.0 110.0
136.0 100.0 192.0  86.0  87.0 113.0  84.0 176.0  95.0  92.0
143.0 147.0  86.0 128.3 211.0  92.2 110.0 105.0  99.0  92.0
 86.0  97.0 161.0 204.0  87.0  95.0 102.0 286.0  93.7 117.0
 88.1 138.2 130.0  94.0  95.0 480.0  98.0  78.0  91.0 104.0
 84.0 105.0  76.0  88.0  77.0 121.0  99.0  85.0  79.0  74.0
100.0 184.0 198.0  78.0 102.0  70.0 270.0 101.0  98.0  88.0
 82.0  96.0  90.0  83.0 370.0  79.0  98.0  98.0 162.0  94.3
 69.0  87.0  78.0 278.0  87.0  84.0  83.0  80.0  87.0 145.0
144.0  79.0  94.0  89.0 177.9 101.0 112.0 161.0 105.0  73.0
 79.0
#-----------------------------------------------------------


cdf_sinWi <- function(par,x){
alpha = par[1]
theta = par[2]
sin((pi/2)*(exp(-alpha*x^(-theta))))
}

pdf_sinWi <- function(par,x){
alpha = par[1]
theta = par[2]
((pi/2))*(alpha*theta*x^(-theta-1)*
exp(-alpha*x^(-theta)))*cos((pi/2)*
```

```
(exp(-alpha*x^(-theta))))
}
goodness.fit(pdf=pdf_sinWi,cdf=cdf_sinWi,
starts=c(1, 1), data = x, method="N",
domain=c(0,1),mle=NULL)



SinIW_pdf=function(x,alpha,theta){
((pi/2))*(alpha*theta*x^(-theta-1)*
exp(-alpha*x^(-theta)))*cos((pi/2)*
(exp(-alpha*x^(-theta))))
}
#Graficos
hist(x,probability=T, ylim=c(0,0.015),col="white",xlab="Gj0",
main="",ylab="pdf")
curve(SinIW_pdf(x,199371.1, 2.595201),col="blue",lwd=2,add=T)



SinIW_Cdf=function(x,alpha,theta){
sin((pi/2)*(exp(-alpha*x^(-theta))))
}

Wexp_Cdf=function(x,alpha,k,lambda){
(1-exp(-(x/lambda)^k))^alpha
}

Beta_ExpCdf=function(x,a,b,lamb){
pbeta((1-exp(-lamb*x)),a,b)
```

```
}

W_Cdf=function(x,alpha,lambda){
(1-exp(-(lambda*x)^alpha))
}




W=function(x,alpha,lambda){
alpha*lambda^(alpha)*x^(alpha-1)
*exp(-(lambda*x)^alpha)
}

BetaExpPdf=function(x,a,b,lambda){
(lambda/beta(a,b))*exp(-lambda*x)*
(1-exp(-lambda*x))^(a-1)*(exp(-lambda*x))
^(b-1)
}

WexpPdf=function(x,alpha,k,lambda){
(k*alpha/lambda)*(x/lambda)^(k-1)*
exp(-(x/lambda)^k)*((1-exp(-(x/lambda)^k))
^(alpha-1))
}

# Weibull Exp- pdf distribution function.

pdf_Wexp <- function(par,x){
alpha  = par[1]
k      = par[2]
lamb   = par[3]
```

```
(k*alpha/lamb)*(x/lamb)^(k-1)*
exp(-(x/lamb)^k)*((1-exp(-(x/lamb)^k))^(alpha-1))
}

# weibull Exp- Cumulative distribution function.
cdf_Wexp <- function(par,x){
alpha  = par[1]
k      = par[2]
lamb   = par[3]
(1-exp(-(x/lamb)^k))^alpha
}
goodness.fit(pdf=pdf_Wexp, cdf=cdf_Wexp,
starts = c(1, 1, 1), data = x, method="S",
domain=c(0,1),mle=NULL)

#---------------------------------------------------
# Beta Exp- pdf distribution function.

pdf_Bexp <- function(par,x){
a      = par[1]
b      = par[2]
lamb   = par[3]
(lamb/beta(a,b))*exp(-lamb*x)*
(1-exp(-lamb*x))^(a-1)*(exp(-lamb*x))^(b-1)
}

# Beta Exp- Cumulative distribution function.
cdf_Bexp <- function(par,x){
a      = par[1]
b      = par[2]
lamb   = par[3]
pbeta((1-exp(-lamb*x)),a,b)
```

```
}
goodness.fit(pdf=pdf_Bexp, cdf=cdf_Bexp,
 starts = c(1, 1, 1), data = x, method="S",
  domain=c(0,1),mle=NULL)


#------------------ pdf distribution function.

pdf_W <- function(par,x){
alpha      = par[1]
lambda     = par[2]
alpha*lambda^(alpha)*x^(alpha-1)
*exp(-(lambda*x)^alpha)
}

# Weibull- Cumulative distribution function.
cdf_W <- function(par,x){
alpha      = par[1]
lambda     = par[2]
(1-exp(-(lambda*x)^alpha))
}
goodness.fit(pdf=pdf_W, cdf=cdf_W,
starts = c(0.1,0.1),
data = x, method="N", domain=c(0,1)
,mle=NULL)



hist(x,probability=T, ylim=c(0,0.015)
,col="white",
xlab="Glycemia",main="",ylab="Density")
curve(SinIW_pdf(x,197058.9,2.592219),
```

```
lty=1,col="black",lwd=2,add=T)
curve(WexpPdf(x,25.6228020,  0.8244303,
23.2143992),
col="red",lwd=2,lty=2,add=T)
curve(BetaExpPdf(x,4.31470510, 1.40734041,
0.01382473),
col="brown",lwd=2,lty=3,add=T)
curve(W(x,1.978864763, 0.007200676),lty=4,col="blue",lwd=2,add=T)
legend('top', c(expression(
list(Sin-IW),
list(W-Exp),
list(B-Exp),
list(W)
)),ncol=1,bty="n",col=c("black","red","brown",
 "blue"),lty=c(1,2,3,4),cex =0.8,lwd=2)


plot.ecdf(x, col="violet",ylim=c(0,1),
xlab="Glycemia",main="",ylab="Cumulative")
curve(SinIW_Cdf(x,197058.9,2.592219),lty=1,
col="black",lwd=2,add=T)
curve(Wexp_Cdf(x,25.6228020,  0.8244303,
23.2143992),
col="red",lwd=2,lty=2,add=T)
curve(Beta_ExpCdf(x,4.31470510, 1.40734041,
0.01382473),
col="brown",lwd=2,lty=3,add=T)
curve(W_Cdf(x,1.978864763, 0.007200676),lty=4,col="blue",lwd=2,add=T)
legend('bottom', c(expression(
list(Sin-IW),
list(W-Exp),
list(B-Exp),
```

```
list(W)
)),ncol=1,bty="n",col=c("black","red",
"brown", "blue"),
lty=c(1,2,3,4),cex =0.8,lwd=2)



##########################################################################
#                       FIM DA DISTRIBUICAO SIN-W                        #
##########################################################################




##########################################################################
#                      INICIO DA DISTRIBUICAO COS-W                      #
##########################################################################

#------------------------------PDF----------------------------------------

par(pty="s")
require(pracma)
require(AdequacyModel)

CosW_pdf=function(x,alpha,lambda){#PDF
((pi/2))*((alpha*lambda^(alpha)*
x^(alpha-1))*exp(-(lambda*x)^alpha))
*sin((pi/2)*((1-exp(-(lambda*x)^alpha))))
}

x = seq(0, 5, length = 1000)
alpha =1.0;  lambda=1.0
```

```
alpha1=1.5;  lambda1=2.0

alpha2=1.0;  lambda2=0.5

alpha3=2.0;  lambda3=2.0

alpha4=0.5;  lambda4=1.0

alpha5=2.0;  lambda5=1.0

curve(CosW_pdf(x,alpha,lambda),
lwd=2,ylim=c(0,2),type="l",cex.lab=0.9,
cex.main=0.5,cex.axis=0.9,lty=1,
from=0,to=5,xlab="x",ylab="Density")

G1=CosW_pdf(x,alpha1,lambda1)
G2=CosW_pdf(x,alpha2,lambda2)
G3=CosW_pdf(x,alpha3,lambda3)
G4=CosW_pdf(x,alpha4,lambda4)
G5=CosW_pdf(x,alpha5,lambda5)

lines(x,G1,lty=2,  lwd=2)
lines(x,G2,lty=3,  lwd=2)
lines(x,G3,lty=4,  lwd=2)
lines(x,G4,lty=5,  lwd=2)

legend('topright',c(expression(
list(alpha==1.0, lambda==1.0),
list(alpha==1.5, lambda==2.0),
list(alpha==1.0, lambda==0.5),
list(alpha==2.0, lambda==2.0),
list(alpha==0.5, lambda==1.0)
)),ncol=1,bty="n",lty=c(1,2,3,4,5),
cex =0.7,lwd=2)

#---------------------CDF----------------------
```

```
CosW_cdf=function(x,alpha,lambda){ #CDF
1-cos((pi/2)*((1-exp(-(lambda*x)^alpha))))
}


x = seq(0, 5, length = 1000)
alpha =1.0;  lambda=1.0
alpha1=1.5;  lambda1=2.0
alpha2=1.0;  lambda2=0.5
alpha3=2.0;  lambda3=2.0
alpha4=0.5;  lambda4=1.0
alpha5=2.0;  lambda5=1.0

curve(CosW_cdf(x,alpha,lambda),
lwd=2,ylim=c(0,1),
type="l",cex.lab=0.9,cex.main=0.5,
cex.axis=0.9,lty=1,from=0,to=5,
xlab="x",ylab="Cumulative")

G1=CosW_cdf(x,alpha1,lambda1)
G2=CosW_cdf(x,alpha2,lambda2)
G3=CosW_cdf(x,alpha3,lambda3)
G4=CosW_cdf(x,alpha4,lambda4)

lines(x,G1,lty=2,   lwd=2)
lines(x,G2,lty=3,  lwd=2)
lines(x,G3,lty=4,  lwd=2)
lines(x,G4,lty=5, lwd=2)

legend('bottomright',c(expression(
list(alpha==1.0,  lambda==1.0),
```

```
list(alpha==1.5,  lambda==2.0),
list(alpha==1.0,  lambda==0.5),
list(alpha==2.0,  lambda==2.0),
list(alpha==0.5,  lambda==1.0)
)),ncol=1,bty="n",lty=c(1,2,3,4,5),
cex =0.7,lwd=2)

#-------------------------------HRF----------------------

RcosW=function(x,alpha,lambda){#Risco
((pi/2))*((alpha*lambda^(alpha)*
x^(alpha-1))*exp(-(lambda*x)^alpha))*
sin((pi/2)*((1-exp(-(lambda*x)^alpha))))
/(cos((pi/2)*((1-exp(-(lambda*x)^alpha)))))
}

x = seq(0, 5, length = 1000)
alpha =0.5;  lambda=2.5
alpha1=0.2;  lambda1=2.5
alpha2=1.0;  lambda2=0.5
alpha3=1.0;  lambda3=0.2
alpha4=2.0;  lambda4=0.5

curve(RcosW(x,alpha,lambda),
lwd=2,ylim=c(0,1),
type="l",cex.lab=0.9,cex.main=0.5,
cex.axis=0.9,lty=1,from=0,to=5,xlab="x",
ylab="Hazard")
G1=RcosW(x,alpha1,lambda1)
G2=RcosW(x,alpha2,lambda2)
G3=RcosW(x,alpha3,lambda3)
G4=RcosW(x,alpha4,lambda4)
```

```
lines(x,G1,lty=2,  lwd=2)
lines(x,G2,lty=3,  lwd=2)
lines(x,G3,lty=4,  lwd=2)
lines(x,G4,lty=5,  lwd=2)

legend('topright',c(expression(
list(alpha==0.5, lambda==2.5),
list(alpha==0.2, lambda==2.5),
list(alpha==1.0, lambda==0.5),
list(alpha==1.0, lambda==0.2),
list(alpha==2.0, lambda==0.5)
)),ncol=1,bty="n",lty=c(1,2,3,4,5),
cex =0.7,lwd=2)


#-------------------------------SURVIVAL-------------------

ScosW=function(x,alpha,lambda){ #Sobrevivencia
cos((pi/2)*((1-exp(-(lambda*x)^alpha))))
}

x = seq(0, 5, length = 1000)
alpha =1.0;  lambda=1.0
alpha1=1.5;  lambda1=1.0
alpha2=1.0;  lambda2=0.5
alpha3=1.0;  lambda3=0.2
alpha4=2.0;  lambda4=0.5

curve(ScosW(x,alpha,lambda),
lwd=2,ylim=c(0,1),
type="l",cex.lab=0.9,cex.main=0.5,
```

```
cex.axis=0.9,lty=1,from=0,to=5,
xlab="x",
ylab="Survival")

G1=ScosW(x,alpha1,lambda1)
G2=ScosW(x,alpha2,lambda2)
G3=ScosW(x,alpha3,lambda3)
G4=ScosW(x,alpha4,lambda4)

lines(x,G1,lty=2,   lwd=2)
lines(x,G2,lty=3,  lwd=2)
lines(x,G3,lty=4,  lwd=2)
lines(x,G4,lty=5, lwd=2)

legend('topright',c(expression(
list(alpha==1.0, lambda==1.0),
list(alpha==1.5, lambda==2.0),
list(alpha==1.0, lambda==0.5),
list(alpha==1.0, lambda==0.2),
list(alpha==1.0, lambda==0.5)
)),ncol=1,bty="n",lty=c(1,2,3,4,5),
cex =0.7,lwd=2)

#----------------------------------------------------

#----------------------------------------------------

require(stats)
require(pracma)
require(AdequacyModel)
x=c(83,51,87,60,28,95,8,27,15,10,18,16,29,54,91,8,17,55,10,35,
47,77,36,17,21,36,18,40,10,7,34,27,28,56,8,25,68,146,89,18,73,
```

```
69,9,37,10,82,29,8,60,61,61,18,169,25,8,26,11,83,11,42,17,14,9,12)

#            Cos Weibull - pdf distribution function.


pdf_cosW <- function(par,x){
alpha      = par[1]
lambda     = par[2]
((pi/2))*((alpha*lambda^(alpha)*x
^(alpha-1))*exp(-(lambda*x)^alpha))
*sin((pi/2)*((1-exp(-(lambda*x)^alpha))))
}

#         Cos Weibull- Cumulative distribution function.
cdf_cosW <- function(par,x){
alpha      = par[1]
lambda     = par[2]
1-cos((pi/2)*((1-exp(-(lambda*x)^alpha))))
}
goodness.fit(pdf=pdf_cosW, cdf=cdf_cosW,
starts =c(0.93, 0.04), data = x, method="S",
 domain=c(0,Inf),mle=NULL)
#-------------------------------------------------------------



# Weibull - pdf distribution function.
pdf_W <- function(par,x){
alpha      = par[1]
lambda     = par[2]
(alpha*lambda^(alpha)*x^(alpha-1))*exp(-(lambda*x)^alpha)
}
# Weibull- Cumulative distribution function.
cdf_W <- function(par,x){
```

```
alpha      = par[1]
lambda     = par[2]
(1-exp(-(lambda*x)^alpha))
}
goodness.fit(pdf=pdf_W, cdf=cdf_W,
starts = c(1, 1), data = x, method="N", domain=c(0,1),mle=NULL)


#------EXPONENTIAL EXPONENTIATED-----------

require(pracma)
require(AdequacyModel)
# EE - pdf distribution function.
pdf_EE <- function(par,x){
alpha      = par[1]
lambda     = par[2]
(alpha*lambda)*exp(-(lambda*x))
*(1-(exp(-(lambda*x))))^(alpha-1)
}
# EE- Cumulative distribution function.
cdf_EE <- function(par,x){
alpha      = par[1]
lambda     = par[2]
(1-exp(-(x*lambda)))^alpha
}
goodness.fit(pdf=pdf_EE, cdf=cdf_EE,
starts =c(1,1), data = x,
method="S", domain=c(0,Inf),mle=NULL)
#-------------------------------------------

pdf_W=function(x,alpha,lambda){
(alpha*lambda^(alpha)*x^(alpha-1))*
```

```
exp(-(lambda*x)^alpha)
}

pdf_EE=function(x,alpha,lambda){
(alpha*lambda)*exp(-(lambda*x))*
(1-(exp(-(lambda*x))))^(alpha-1)
}

pdf_cosW=function(x,alpha,lambda){
((pi/2))*((alpha*lambda^(alpha)*
x^(alpha-1))*exp(-(lambda*x)^alpha))
*sin((pi/2)*((1-exp(-(lambda*x)^alpha))))
}

par(pty="s")
hist(x,probability=T, col="white",ylim=c(0,0.025),cex.lab=0.9
,cex.axis=0.9,xlab="Time",main="(a)",
ylab="Density")
curve(pdf_W(x,1.27459359, 0.02312992),
lty = 3,lwd=2,col="blue",add=T)
curve(pdf_EE(x,1.74593848, 0.03543284),
lty = 2,lwd=2,col="red",add=T)
curve(pdf_cosW(x, 0.93885070, 0.03599627),
cex.main=0.9,lty = 1,lwd=2,col="black",add=T)
legend('top', c(expression(
list(W),
list(EE),
list(Cos-W)
)),ncol=1,bty="n",col=c("blue","red","black"),
lty=c(3,2,1),cex =0.7,lwd=2)
```

```
#-----------------------------------------------

cdf_cosW=function(x,alpha,lambda){
1-cos((pi/2)*(1-exp(-(lambda*x)^alpha)))
}

cdf_EE=function(x,alpha,lambda){
(1-exp(-(x*lambda)))^alpha
}

cdf_W=function(x,alpha,lambda){
(1-exp(-(lambda*x)^alpha))
}


require(stats)
plot.ecdf(x, col="violet",ylim=c(0,1),
cex.lab=0.9,cex.axis=0.9,xlab="Time",
main="(b)",ylab="Cumulative")
curve(cdf_W(x,1.27459359, 0.02312992),
lty = 3,
lwd=2,col="blue",add=T)
curve(cdf_EE(x,1.74593848, 0.03543284),
lty = 2,lwd=2,
col="red",add=T)
curve(cdf_cosW(x,0.93885070, 0.03599627),
lty = 1,lwd=2,
col="black",add=T)
legend('bottom', c(expression(
list(W),
list(EE),
list(Cos-W)
```

```
)),ncol=1,bty="n",col=c("blue","red","black"),
lty=c(3,2,1),cex =0.7,lwd=2)


###########################################################################
#                       FIM DA DISTRIBUICAO COS-W                         #
###########################################################################




###########################################################################
#                      INICIO DA DISTRIBUICAO TAN-BXII                    #
###########################################################################

#-------------------PDF--------------------

#Tan-BurrXII
require(pracma)
par(pty="s")
pdf_TanB=function(x,c,k,s){
((pi/4))*(c*k*x^(c-1)*s^(-c)*
(1+(x/s)^c)^(-k-1))*sec((pi/4)
*(1-(1+(x/s)^c)^(-k)))^2
}

x = seq(0, 5, length = 1000)
c =2.5;   k=3.0;   s=3.5;
c1=2.0;   k1=3.0;  s1=3.0;
c2=2.5;   k2=1.0;  s2=1.0;
c3=3.0;   k3=1.0;  s3=1.0;
c4=3.5;   k4=1.0;  s4=1.0;
```

```
c5=4.0;   k5=1.5;  s5=2.0;

curve(pdf_TanB(x,c,k,s),lty=1,
cex.lab=0.9,cex.main=0.9,cex.axis=0.9,
lwd=2,ylim=c(0,1.0),type="l",
from=0,to=5,xlab="x",ylab="Density",
main=" ")

G1=pdf_TanB(x,c1,k1,s1)
G2=pdf_TanB(x,c2,k2,s2)
G3=pdf_TanB(x,c3,k3,s3)
G4=pdf_TanB(x,c4,k4,s4)
G5=pdf_TanB(x,c5,k5,s5)

lines(x,G1,lty=2,lwd=2)
lines(x,G2,lty=3,lwd=2)
lines(x,G3,lty=4,lwd=2)
lines(x,G4,lty=5,lwd=2)
lines(x,G5,lty=6,lwd=2)

legend('topright', c(expression(
list(c==2.5, k==3.0, s==3.5),
list(c==2.0, k==3.0, s==3.0),
list(c==2.5, k==1.0, s==1.0),
list(c==3.0, k==1.0, s==1.0),
list(c==3.5, k==1.0, s==1.0),
list(c==4.0, k==1.5, s==2.0)
)),ncol=1,bty="n",lty=c(1,2,3,4,5,6),
cex =0.7,lwd=2)


#----------------CDF----------------------
```

```
par(pty="s")
cdf_TanB=function(x,c,k,s){#CDF
tan((pi/4)*(1-(1+(x/s)^c)^(-k)))
}

x = seq(0, 5, length = 1000)
c =1.0;   k=1.0;   s=1.0;
c1=2.0;   k1=1.0;  s1=1.0;
c2=2.5;   k2=1.0;  s2=1.0;
c3=3.0;   k3=1.0;  s3=1.0;
c4=3.5;   k4=1.0;  s4=1.0;
c5=4.0;   k5=1.0;  s5=1.0;

curve(cdf_TanB(x,c,k,s),lty=1,
cex.lab=0.9,cex.main=0.9,cex.axis=0.9,
lwd=2,ylim=c(0,1.0),type="l",
from=0,to=5,xlab="x",
ylab="Cumulative",main=" ")

G1=cdf_TanB(x,c1,k1,s1)
G2=cdf_TanB(x,c2,k2,s2)
G3=cdf_TanB(x,c3,k3,s3)
G4=cdf_TanB(x,c4,k4,s4)
G5=cdf_TanB(x,c5,k5,s5)

lines(x,G1,lty=2, lwd=2)
lines(x,G2,lty=3, lwd=2)
lines(x,G3,lty=4, lwd=2)
lines(x,G4,lty=5, lwd=2)
lines(x,G5,lty=6, lwd=2)

legend('bottomright', c(expression(
```

```
list(c==1.0, k==1.0, s==1.0),
list(c==2.0, k==1.0, s==1.0),
list(c==2.5, k==1.0, s==1.0),
list(c==3.0, k==1.0, s==1.0),
list(c==3.5, k==1.0, s==1.0),
list(c==4.0, k==1.0, s==1.0)
)),ncol=1,bty="n",
lty=c(1,2,3,4,5,6),cex =0.7,lwd=2)


#------------- Risco Unimodal -------------

par(pty="s")
hrf_TanB=function(x,c,k,s){
(((pi/4))*(c*k*x^(c-1)*s^(-c)
*(1+(x/s)^c)^(-k-1))*sec((pi/4)
*(1-(1+(x/s)^c)^(-k)))^2)/
(1-tan((pi/4)*(1-(1+(x/s)^c)^(-k))))
}

x = seq(0, 5, length = 1000)
c =1.5;   k=0.9;   s=0.8;
c1=2.2;   k1=3.4;  s1=2.5;
c2=2.5;   k2=1.5;  s2=1.9;
c3=3.0;   k3=1.2;  s3=1.3;
c4=5.5;   k4=0.1;  s4=1.0;
c5=4.5;   k5=0.8;  s5=1.0;

curve(hrf_TanB(x,c,k,s),
cex.lab=0.9,cex.main=0.9,
cex.axis=0.9,lty=1,lwd=2,
ylim=c(0,2.0),type="l",from=0,to=5,
```

```
xlab="x",ylab="Hazard",main=" ")

G1=hrf_TanB(x,c1,k1,s1)
G2=hrf_TanB(x,c2,k2,s2)
G3=hrf_TanB(x,c3,k3,s3)
G4=hrf_TanB(x,c4,k4,s4)
G5=hrf_TanB(x,c5,k5,s5)

lines(x,G1,lty=2, lwd=2)
lines(x,G2,lty=3, lwd=2)
lines(x,G3,lty=4, lwd=2)
lines(x,G4,lty=5, lwd=2)
lines(x,G5,lty=6, lwd=2)

legend('topright', c(expression(
list(c==1.5, k==0.9, s==0.8),
list(c==2.2, k==3.4, s==2.5),
list(c==2.5, k==1.5, s==1.9),
list(c==3.0, k==1.2, s==1.3),
list(c==5.5, k==0.1, s==1.0),
list(c==4.5, k==0.8, s==1.0)
)),ncol=1,bty="n",lty=c(1:6),
cex =0.7,lwd=2)

#==========Hazard increasing ============
par(pty="s")
hrf_TanB=function(x,c,k,s){
(((pi/4))*(c*k*x^(c-1)*s^(-c)*
(1+(x/s)^c)^(-k-1))*sec((pi/4)*
(1-(1+(x/s)^c)^(-k)))^2)/
(1-tan((pi/4)*(1-(1+(x/s)^c)^(-k))))
}
```

```
x = seq(0, 5, length = 1000)
c =4.0;   k=2.4;   s=4.0;
c1=4.0;   k1=2.8;  s1=5.0;
c2=3.0;   k2=3.4;  s2=7.0;
c3=1.5;   k3=5.5;  s3=4.0;
c4=2.5;   k4=5.0;  s4=5.5;
c5=5.0;   k5=2.5;  s5=3.0;

#a=2; b=2; s=7; k=1.4; c=5
#a=2; b=1.5; s=5; k=1.4; c=4

curve(hrf_TanB(x,c,k,s),cex.lab=0.9,
cex.main=0.9,cex.axis=0.9,lty=1,lwd=2,
ylim=c(0,2.0),type="l",from=0,to=5,
xlab="x",ylab="Hazard",main=" ")

G1=hrf_TanB(x,c1,k1,s1)
G2=hrf_TanB(x,c2,k2,s2)
G3=hrf_TanB(x,c3,k3,s3)
G4=hrf_TanB(x,c4,k4,s4)
G5=hrf_TanB(x,c5,k5,s5)

lines(x,G1,lty=2, lwd=2)
lines(x,G2,lty=3, lwd=2)
lines(x,G3,lty=4, lwd=2)
lines(x,G4,lty=5, lwd=2)
lines(x,G5,lty=6, lwd=2)

legend('topleft', c(expression(
list(c==4.0, k==2.4, s==4.0),
list(c==4.0, k==2.8, s==5.0),
```

```
list(c==3.0, k==3.4, s==7.0),
list(c==1.5, k==5.5, s==4.0),
list(c==2.5, k==5.0, s==5.5),
list(c==5.0, k==2.5, s==3.0)
)),ncol=1,bty="n",lty=c(1:6),
cex =0.7,lwd=2)

#==============Hazard decreasing===========



par(pty="s")
hrf_TanB=function(x,c,k,s){
(((pi/4))*(c*k*x^(c-1)*s^(-c)*
(1+(x/s)^c)^(-k-1))*sec((pi/4)*
(1-(1+(x/s)^c)^(-k)))^2)/
(1-tan((pi/4)*(1-(1+(x/s)^c)^(-k))))
}

x = seq(0, 5, length = 1000)
c =0.2;   k=1.4;   s=1.2;
c1=0.5;   k1=1.8;  s1=5.0;
c2=0.3;   k2=3.4;  s2=7.0;
c3=0.5;   k3=5.5;  s3=4.0;
c4=0.5;   k4=5.0;  s4=5.5;
c5=0.2;   k5=2.5;  s5=3.0;

curve(hrf_TanB(x,c,k,s),cex.lab=0.9,
cex.main=0.9,cex.axis=0.9,lty=1,lwd=2,
ylim=c(0,2.0),type="l",from=0,to=5,
xlab="x",ylab="Hazard",main=" ")

G1=hrf_TanB(x,c1,k1,s1)
```

```
G2=hrf_TanB(x,c2,k2,s2)
G3=hrf_TanB(x,c3,k3,s3)
G4=hrf_TanB(x,c4,k4,s4)
G5=hrf_TanB(x,c5,k5,s5)

lines(x,G1,lty=2, lwd=2)
lines(x,G2,lty=3, lwd=2)
lines(x,G3,lty=4, lwd=2)
lines(x,G4,lty=5, lwd=2)
lines(x,G5,lty=6, lwd=2)

legend('topright', c(expression(
list(c==0.2, k==1.4, s==1.2),
list(c==0.5, k==1.8, s==5.0),
list(c==0.3, k==3.4, s==7.0),
list(c==0.5, k==5.5, s==4.0),
list(c==0.5, k==5.0, s==5.5),
list(c==0.2, k==2.5, s==3.0)
)),ncol=1,bty="n",lty=c(1:6),
cex =0.7,lwd=2)



#--------------------SURVIVAL---------------
par(pty="s")
surv_TanB=function(x,c,k,s){
1-tan((pi/4)*(1-(1+(x/s)^c)^(-k)))
}

x = seq(0, 5, length = 1000)
c =1.0;   k=1.0;   s=1.0;
c1=2.0;   k1=1.0;  s1=1.0;
```

```
c2=2.5;   k2=1.0;  s2=1.0;
c3=3.0;   k3=1.0;  s3=1.0;
c4=3.5;   k4=1.0;  s4=1.0;
c5=4.0;   k5=1.0;  s5=1.0;

curve(surv_TanB(x,c,k,s),lty=1,
cex.lab=0.9,cex.main=0.9,cex.axis=0.9,
lwd=2,ylim=c(0,1.0),type="l",from=0,to=5,
xlab="x",ylab="Survival",main=" ")

G1=surv_TanB(x,c1,k1,s1)
G2=surv_TanB(x,c2,k2,s2)
G3=surv_TanB(x,c3,k3,s3)
G4=surv_TanB(x,c4,k4,s4)
G5=surv_TanB(x,c5,k5,s5)

lines(x,G1,lty=2, lwd=2)
lines(x,G2,lty=3, lwd=2)
lines(x,G3,lty=4, lwd=2)
lines(x,G4,lty=5, lwd=2)
lines(x,G5,lty=6, lwd=2)

legend('topright', c(expression(
list(c==1.0, k==1.0, s==1.0),
list(c==2.0, k==1.0, s==1.0),
list(c==2.5, k==1.0, s==1.0),
list(c==3.0, k==1.0, s==1.0),
list(c==3.5, k==1.0, s==1.0),
list(c==4.0, k==1.0, s==1.0)
)),ncol=1,bty="n",lty=c(1:6),
cex =0.7,lwd=2)
```

```
##################################################################

x=c(0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309,
    1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912,
    2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661,
    3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035,
    1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432,
    2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154,
    2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103,
    4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485,
    1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,
    2.324, 3.376, 4.663)

##################################################################

#-------------------PDF----------------------------

pdf_KumW=function(x,a,b,c,lambda){#CDF
a*b*c*lambda^(c)*x^(c-1)*exp(-(lambda*x)^c)*
(1-exp(-(lambda*x)^c))^(a-1)*
(1-(1-exp(-(lambda*x)^c))^a)^(b-1)
}

pdf_Burr=function(x,c,k,s){#CDF
c*k*x^(c-1)*s^(-c)*(1+(x/s)^c)^(-k-1)
}

pdf_KumB=function(x,a,b,c,k,s){#CDF
a*b*c*k*s^(-c)*x^(c-1)*(1+(x/s)^c)
^(-k-1)*
(1-(1+(x/s)^c)^(-k))^(a-1)*
```

```
(1-(1-(1+(x/s)^c)^(-k))^a)^(b-1)
}

pdf_TanB=function(x,c,k,s){#CDF
((pi/4))*(c*k*x^(c-1)*s^(-c)*
(1+(x/s)^c)^(-k-1))*sec((pi/4)*
(1-(1+(x/s)^c)^(-k)))^2
}

#-------------------CDF-------------------------------------

cdf_KumW=function(x,a,b,c,lambda){#CDF
1-(1-(1-exp(-(lambda*x)^c))^a)^b
}

cdf_Burr=function(x,c,k,s){#CDF
1-(1+(x/s)^c)^(-k)
}

cdf_KumB=function(x,a,b,c,k,s){#CDF
1-(1-(1-(1+(x/s)^c)^(-k))^a)^b
}

cdf_TanB=function(x,c,k,s){#CDF
tan((pi/4)*(1-(1+(x/s)^c)^(-k)))
}




# KumWeibull - pdf distribution function.

pdf_KumW <- function(par,x){
```

```
a      = par[1]
b      = par[2]
c      = par[3]
lambda = par[4]
a*b*c*lambda^(c)*x^(c-1)*
exp(-(lambda*x)^c)*
(1-exp(-(lambda*x)^c))^(a-1)*
(1-(1-exp(-(lambda*x)^c))^a)^(b-1)
}

# KumWeibull- Cumulative distribution function.
cdf_KumW <- function(par,x){
a      = par[1]
b      = par[2]
c      = par[3]
lambda = par[4]
1-(1-(1-exp(-(lambda*x)^c))^a)^b
}
goodness.fit(pdf=pdf_KumW, cdf=cdf_KumW,
starts =c(1,1,1,1), data = x, method="S",
domain=c(0,Inf),mle=NULL)


#-----------------------------------------------------
# Burr - pdf distribution function.

pdf_Burr <- function(par,x){
c      = par[1]
k      = par[2]
s      = par[3]
c*k*x^(c-1)*s^(-c)*(1+(x/s)^c)^(-k-1)
}
```

```
# Burr- Cumulative distribution function.
cdf_Burr <- function(par,x){
c      = par[1]
k      = par[2]
s      = par[3]
1-(1+(x/s)^c)^(-k)
}
goodness.fit(pdf=pdf_Burr, cdf=cdf_Burr,
starts =c(1,1,1), data = x, method="S",
domain=c(0,Inf),mle=NULL)


#-----------------------------------------------------

# Kum Burr - pdf distribution function.

pdf_KumB <- function(par,x){
a      = par[1]
b      = par[2]
c      = par[3]
k      = par[4]
s      = par[5]
a*b*c*k*s^(-c)*x^(c-1)*(1+(x/s)^c)
^(-k-1)*(1-(1+(x/s)^c)^(-k))^(a-1)
*(1-(1-(1+(x/s)^c)^(-k))^a)^(b-1)
}

# Kum Burr- Cumulative distribution function.
cdf_KumB <- function(par,x){
a      = par[1]
b      = par[2]
c      = par[3]
```

```
k      = par[4]
s      = par[5]
1-(1-(1-(1+(x/s)^c)^(-k))^a)^b
}
goodness.fit(pdf=pdf_KumB, cdf=cdf_KumB,
starts =c(1,1,1,1,1), data = x, method="S",
 domain=c(0,Inf),mle=NULL)

#---------------------------------------------------------------

# Tan Burr - pdf distribution function.

pdf_TanB <- function(par,x){
c      = par[1]
k      = par[2]
s      = par[3]
((pi/4))*(c*k*x^(c-1)*s^(-c)*
(1+(x/s)^c)^(-k-1))*sec((pi/4)
*(1-(1+(x/s)^c)^(-k)))^2
}

# Tan Burr- Cumulative distribution function.
cdf_TanB <- function(par,x){
c      = par[1]
k      = par[2]
s      = par[3]
tan((pi/4)*(1-(1+(x/s)^c)^(-k)))
}
goodness.fit(pdf=pdf_TanB, cdf=cdf_TanB,
starts =c(1,1,1), data = x, method="S",
domain=c(0,Inf),mle=NULL)
```

```
###################################################
#                                                                           #
#  CONSTRUCAO DA DENSIDADE E DA ACUMULADA
#  PARA O BANCO DE DADOS
#                                                                           #
###################################################

par(pty="s")
hist(x,probability=T, col="white",
ylim=c(0,0.5),xlab="x",
main="",ylab="Density")
curve(pdf_TanB(x, 2.326198, 22.960152,
9.899044),lty = 1, lwd=2,col="black",add=T)
curve(pdf_KumB(x, 0.3014036, 2.3496068,
6.6348990, 5.6576604, 6.5356078),lty = 2,
lwd=2,col="red",add=T)
curve(pdf_Burr(x, 2.477123, 14.552212,
8.335620),lty = 3,lwd=2,col="blue",add=T)
curve(pdf_KumW(x, 0.4069757, 0.2389082,
3.3656304, 0.4729780),lty = 4,lwd=2,
col="brown",add=T)
legend('top', c(expression(
list(Tan-B),
list(Kum-B),
list(Burr),
list(KumW)
)),ncol=1,bty="n",lty=c(1,2,3,4),
col=c("black","red","blue","brown"),
cex =0.8,lwd=2)

par(pty="s")
```

```
require(stats)
plot.ecdf(x, col="violet",ylim=c(0,1),
xlab="x",main="",ylab="Cumulative")
curve(cdf_TanB(x, 2.326198, 22.960152,
9.899044),lty = 1, lwd=2,col="black",add=T)
curve(cdf_KumB(x, 0.3014036, 2.3496068,
6.6348990, 5.6576604, 6.5356078),lty = 2,
lwd=2,col="red",add=T)
curve(cdf_Burr(x, 2.477123, 14.552212,
8.335620),lty = 3,lwd=2,col="blue",add=T)
curve(cdf_KumW(x, 0.4069757, 0.2389082,
3.3656304, 0.4729780),lty = 4,lwd=2,
col="brown",add=T)
legend('topleft', c(expression(
list(Tan-B),
list(Kum-B),
list(Burr),
list(KumW)
)),ncol=1,bty="n",
col=c("black","red","blue","brown"),
lty=c(1,2,3,4),cex =0.8,lwd=2)


###############################################################
#                      FIM DA DISTRIBUICAO TAN-BXII           #
###############################################################


###############################################################
#                     INICIO DA DISTRIBUICAO SEC-KumW         #
###############################################################
```

```
require(pracma)
require(AdequacyModel)
#par(mfrow = c(1,2), pty = "s")
#par(pin = c(5.5, 5.5,5.5, 5.5))

par(pty = "s")
SecKumWpdf=function(x,a,b,c,lambda){
((pi/3))*(a*b*c*lambda^c*x^(c-1)*exp(-(lambda*x)^c)
*(1-exp(-lambda*x)^c)^(a-1)*(1-(1-exp(-lambda*x)^c)^a)
^(b-1))*sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))
*tan((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))
}

x = seq(0, 5, length = 1000)
a =1.0;   b=1.0;   c=1.0;  lambda=1.0;
a1=2.0;   b1=2.0;  c1=1.0; lambda1=1.0;
a2=4.0;   b2=5.0;  c2=1.0; lambda2=1.0;
a3=6.0;   b3=3.5;  c3=1.0; lambda3=1.0;
a4=1.5;   b4=0.8;  c4=1.0; lambda4=1.0;
a5=0.5;   b5=1.0;  c5=1.0; lambda5=1.0;

curve(SecKumWpdf(x,a,b,c,lambda),lwd=2,ylim=c(0,1.0)
,type="l",lty=1,cex.lab=0.9,cex.main=0.9,cex.axis=0.9,
from=0,to=5,
xlab="x",ylab="Density",main="")

G2=SecKumWpdf(x,a1,b1,c1,lambda1)
G3=SecKumWpdf(x,a2,b2,c2,lambda2)
G4=SecKumWpdf(x,a3,b3,c3,lambda3)
G5=SecKumWpdf(x,a4,b4,c4,lambda4)
G6=SecKumWpdf(x,a5,b5,c5,lambda5)
```

```
lines(x,G2,lty=2,lwd=2)
lines(x,G3,lty=3,lwd=2)
lines(x,G4,lty=4,lwd=2)
lines(x,G5,lty=5,lwd=2)
lines(x,G6,lty=6,lwd=2)
legend('topright', c(expression(
list(a==1.0, b==1.0, c==1.0, lambda==1.0),
list(a==2.0, b==2.0, c==1.0, lambda==1.0),
list(a==4.0, b==5.0, c==1.0, lambda==1.0),
list(a==6.0, b==3.5, c==1.0, lambda==1.0),
list(a==1.5, b==0.8, c==1.0, lambda==1.0),
list(a==0.5, b==1.0, c==1.0, lambda==1.0)
)),ncol=1,bty="n",lty=c(1:6),cex =0.7,lwd=2)

#------------------------------CDF---------------

par(pty = "s")
require(stats)
SecKumWcdf=function(x,a,b,c,lambda){
sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))-1
}

x = seq(0, 5, length = 1000)
a =1.0;   b=1.0;   c=1.0;  lambda=1.0;
a1=3.0;   b1=0.5;  c1=2.0; lambda1=2.0;
a2=4.0;   b2=5.0;  c2=1.0; lambda2=1.0;
a3=6.0;   b3=3.5;  c3=1.0; lambda3=1.0;
a4=1.5;   b4=0.8;  c4=1.0; lambda4=1.0;
a5=0.5;   b5=1.0;  c5=1.0; lambda5=1.0;

curve(SecKumWcdf(x,a,b,c,lambda),lty=1,cex.lab=0.9,
cex.main=0.9,cex.axis=0.9,lwd=2,
```

```
ylim=c(0,1.0),type="l",from=0,to=5,xlab="x",
ylab="Cumulative",main=" ")
G1=SecKumWcdf(x,a1,b1,c1,lambda1)
G2=SecKumWcdf(x,a2,b2,c2,lambda2)
G3=SecKumWcdf(x,a3,b3,c3,lambda3)
G4=SecKumWcdf(x,a4,b4,c4,lambda4)
G5=SecKumWcdf(x,a5,b5,c5,lambda5)

lines(x,G1,lty=2, lwd=2)
lines(x,G2,lty=3, lwd=2)
lines(x,G3,lty=4, lwd=2)
lines(x,G4,lty=5, lwd=2)
lines(x,G5,lty=6, lwd=2)

legend('bottomright', c(expression(
list(a==1.0, b==1.0, c==1.0, lambda==1.0),
list(a==3.0, b==0.5, c==2.0, lambda==2.0),
list(a==4.0, b==5.0, c==1.0, lambda==1.0),
list(a==6.0, b==3.5, c==1.0, lambda==1.0),
list(a==1.5, b==0.8, c==1.0, lambda==1.0),
list(a==0.5, b==1.0, c==1.0, lambda==1.0)
)),ncol=1,bty="n",lty=c(1:6),cex =0.7,lwd=2)

#------------------------------HRF-----------------------



#RISK part I bathtub
par(pty = "s")
SecKumWhrf=function(x,a,b,c,lambda){
((pi/3)*(a*b*c*lambda^c*x^(c-1))*exp(-(lambda*x)^c)
*(1-exp(-(lambda*x)^c))^(a-1)*(1-(1-exp(-(lambda*x)^c))^a)
^(b-1))*sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))*
```

```
tan((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))/
(2-sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
#(((a*b*c*lambda^c*x^(c-1))*exp(-(lambda*x)^c)*
(1-exp(-(lambda*x)^c))^(a-1)*(1-(1-exp(-(lambda*x)^c))^a)
^(b-1)))/(1-(1-(1-(1-exp(-(lambda*x)^c))^a)^b))
}
x = seq(0, .6, length = 1000)
a =0.1;   b=1.9;   c =3.6; lambda=2.0;
a1=0.1;   b1=1.7;  c1=3.0; lambda1=2.2;
a2=0.15;  b2=2.0;  c2=2.5; lambda2=2.3;
a3=0.1;   b3=1.7;  c3=3.7; lambda3=2.5;
a4=0.2;   b4=2.1;  c4=2.0; lambda4=2.5;

curve(SecKumWhrf(x,a,b,c,lambda),lty=1,cex.lab=0.9,
cex.main=0.9,cex.axis=0.9,lwd=2.0,ylim=c(0,25),
type="l",from=0,to=.6,xlab="x",
ylab="Hazard",main=" ")

G1=SecKumWhrf(x,a1,b1,c1,lambda1)
G2=SecKumWhrf(x,a2,b2,c2,lambda2)
G3=SecKumWhrf(x,a3,b3,c3,lambda3)
G4=SecKumWhrf(x,a4,b4,c4,lambda4)

lines(x,G1,lty=2,lwd=2)
lines(x,G2,lty=3,lwd=2)
lines(x,G3,lty=4,lwd=2)
lines(x,G4,lty=5,lwd=2)


legend('bottomright', c(expression(
list(a==0.1,  b==1.9, c==3.6, lambda==2.0),
list(a==0.1,  b==1.7, c==3.0, lambda==2.2),
```

```
list(a==0.15, b==2.0, c==2.5, lambda==2.3),
list(a==0.1,  b==1.7, c==3.7, lambda==2.5),
list(a==0.2,  b==2.1, c==2.0, lambda==2.5)
)),ncol=1,bty="n",lty=c(1:6),cex =0.7,lwd=2)

#===================================================

#RISK II Unimodal

par(pty = "s")
SecKumWhrf=function(x,a,b,c,lambda){
((pi/3)*(a*b*c*lambda^c*x^(c-1))*exp(-(lambda*x)^c)
*(1-exp(-(lambda*x)^c))^(a-1)*(1-(1-exp(-(lambda*x)^c))^a)
^(b-1))*sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))*
tan((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))/
(2-sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
#(((a*b*c*lambda^c*x^(c-1))*exp(-(lambda*x)^c)*
(1-exp(-(lambda*x)^c))^(a-1)*
(1-(1-exp(-(lambda*x)^c))^a)^(b-1)))
/(1-(1-(1-(1-exp(-(lambda*x)^c))^a)^b))
}
x = seq(0, 1.5, length = 1000)
a =10;    b=12.0;   c =0.5; lambda=14.0;
a1=12.0;  b1=12.0;  c1=0.6; lambda1=9.0;
a2=8.0;   b2=11.0;  c2=0.6; lambda2=5.0;
a3=11.0;  b3=17.0;  c3=0.7; lambda3=4.0;
a4=45.0;  b4=17.0;  c4=0.4; lambda4=25.0;
curve(SecKumWhrf(x,a,b,c,lambda),lty=1,
cex.lab=0.9,cex.main=0.9,cex.axis=0.9,lwd=2.0,
ylim=c(0,25),type="l",from=0,to=1.5,xlab="x",
ylab="Hazard",main=" ")
G1=SecKumWhrf(x,a1,b1,c1,lambda1)
```

```
G2=SecKumWhrf(x,a2,b2,c2,lambda2)
G3=SecKumWhrf(x,a3,b3,c3,lambda3)
G4=SecKumWhrf(x,a4,b4,c4,lambda4)
lines(x,G1,lty=2,lwd=2)
lines(x,G2,lty=3,lwd=2)
lines(x,G3,lty=4,lwd=2)
lines(x,G4,lty=5,lwd=2)
legend('bottomright', c(expression(
list(a==10,    b==12.0, c==0.5, lambda==14.0),
list(a==12.0,  b==12.0, c==0.6, lambda==9.0),
list(a==8.0,   b==11.0, c==0.6, lambda==5.0),
list(a==11.0,  b==17.0, c==0.7, lambda==4.0),
list(a==45.0,  b==17.0, c==0.4, lambda==25.0)
)),ncol=1,bty="n",lty=c(1:6),cex =0.7,lwd=2)


#=================================================================
#RISK III Increasing and decreasing

par(pty = "s")
SecKumWhrf=function(x,a,b,c,lambda){
((pi/3)*(a*b*c*lambda^c*x^(c-1))*exp(-(lambda*x)^c)
*(1-exp(-(lambda*x)^c))^(a-1)*(1-(1-exp(-(lambda*x)^c))^a)
^(b-1))*sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))*
tan((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))/
(2-sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
#(((a*b*c*lambda^c*x^(c-1))*exp(-(lambda*x)^c)*
(1-exp(-(lambda*x)^c))^(a-1)*(1-(1-exp(-(lambda*x)^c))^a)
^(b-1)))/(1-(1-(1-(1-exp(-(lambda*x)^c))^a)^b))
}
x = seq(0,1.5, length = 1000)
a =0.5;    b=0.4;   c =0.2; lambda=1.0;
```

```
a1=15.0;  b1=3.5;   c1=1.5; lambda1=1.2;
a2=10.0;  b2=4.0;   c2=2.0; lambda2=0.9;
a3=0.5;   b3=0.3;   c3=0.4; lambda3=0.9;
a4=45.0;  b4=3.0;  c4=0.4; lambda4=25.0;
curve(SecKumWhrf(x,a,b,c,lambda),lty=1,cex.lab=0.9
,cex.main=0.9,cex.axis=0.9,lwd=2.0,ylim=c(0,1.5),
type="l",from=0,to=1.5,xlab="x",
ylab="Hazard",main=" ")
G1=SecKumWhrf(x,a1,b1,c1,lambda1)
G2=SecKumWhrf(x,a2,b2,c2,lambda2)
G3=SecKumWhrf(x,a3,b3,c3,lambda3)
G4=SecKumWhrf(x,a4,b4,c4,lambda4)
lines(x,G1,lty=2,lwd=2)
lines(x,G2,lty=3,lwd=2)
lines(x,G3,lty=4,lwd=2)
lines(x,G4,lty=5,lwd=2)
legend('topleft', c(expression(
list(a==0.5,    b==0.4, c==0.2, lambda==1.0),
list(a==15.0,   b==3.5, c==1.5, lambda==1.2),
list(a==10.0,   b==4.0, c==2.0, lambda==0.9),
list(a== 0.5,   b==0.3, c==0.4, lambda==0.9),
list(a==45.0,   b==3.0,c==0.4, lambda==25.0)
)),ncol=1,bty="n",lty=c(1:6),cex =0.7,lwd=2)


#----------------------------SURVIVAL------------------
par(pty = "s")
SecKumWsurv=function(x,a,b,c,lambda){
2-sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b))
}
x = seq(0, 8, length = 1000)
a =1.0;   b=1.0;   c =1.0;  lambda=1.0;
```

```
a1=2.0;   b1=1.0;  c1=1.0; lambda1=0.5;
a2=2.5;   b2=1.0;  c2=1.0; lambda2=0.7;
a3=1.0;   b3=1.0;  c3=1.5; lambda3=1.5;
a4=1.5;   b4=1.0;  c4=2.5; lambda4=0.3;
curve(SecKumWsurv(x,a,b,c,lambda),lty=1,
cex.lab=0.9,cex.main=0.9,cex.axis=0.9,col="black",
lwd=2,ylim=c(0,1.0),type="l",from=0,to=8,xlab="x",
ylab="Survival",main="")
G2=SecKumWsurv(x,a1,b1,c1,lambda1)
G3=SecKumWsurv(x,a2,b2,c2,lambda2)
G4=SecKumWsurv(x,a3,b3,c3,lambda3)
G5=SecKumWsurv(x,a4,b4,c4,lambda4)
lines(x,G2,lty=2,lwd=2)
lines(x,G3,lty=3,lwd=2)
lines(x,G4,lty=4,lwd=2)
lines(x,G5,lty=5,lwd=2)
legend('topright', c(expression(
list(a==1.0, b==1.0, c==1.0, lambda==1.0),
list(a==2.0, b==1.0, c==1.0, lambda==0.5),
list(a==2.5, b==1.0, c==1.0, lambda==0.7),
list(a==1.0, b==1.0, c==1.5, lambda==1.5),
list(a==1.5, b==1.0, c==2.5, lambda==0.3)
)),ncol=1,bty="n",lty=c(1:6),cex =0.7,lwd=2)


#============================================================


############################################################
#                                                          #
#     USO DO PACOTE ADEQUACY MODEL PARA ESTIMATIVAS - MLE  #
#                                                          #
```

```
#                                                           #
#############################################################


#            BANCO DE DADOS - LIFETIMES OF 50 DEVICES


x=c(0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18,
    18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63,
    67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85,
    85, 85, 85, 85, 86, 86)

BetaExpPdf=function(x,a,b,lambda){
(lambda/beta(a,b))*exp(-lambda*x)*
(1-exp(-lambda*x))^(a-1)*(exp(-lambda*x))^(b-1)
}

WexpPdf=function(x,alpha,k,lambda){
(k*alpha/lambda)*(x/lambda)^(k-1)*exp(-(x/lambda)^k)*
((1-exp(-(x/lambda)^k))^(alpha-1))
}

WexpCdf=function(x,alpha,k,lambda){
(1-exp(-(x/lambda)^k))^alpha
}

BetaExpCdf=function(x,a,b,lamb){
pbeta((1-exp(-lamb*x)),a,b)
}

KumW=function(x,a,b,c,lambda){
a*b*c*lambda^(c)*x^(c-1)*exp(-(lambda*x)^c)
*(1-exp(-(lambda*x)^c))
```

```
^(a-1)*(1-(1-exp(-(lambda*x)^c))^a)^(b-1)
}

Wpdf=function(x,alpha,lambda){
(alpha*lambda^(alpha)*x^(alpha-1))*
exp(-(lambda*x)^alpha)
}



SecKumWcdf=function(x,a,b,c,lambda){
(sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))
^a)^b)))-1
}

KumWPcdf=function(x,a,b,c,lambda,beta){
(1 - exp(lambda*(-(1-(1-(1-exp(-(x*beta)
^c))^a)^b))))/(1-exp(-lambda))
}

KumWcdf=function(x,a,b,c,lambda){
1-(1-(1-exp(-(lambda*x)^c))^a)^b
}

Wcdf=function(x,alpha,lambda){
(1-exp(-(lambda*x)^alpha))
}




# Weibull Exp- pdf distribution function.

pdf_Wexp <- function(par,x){
```

```
alpha  = par[1]
k      = par[2]
lamb   = par[3]
(k*alpha/lamb)*(x/lamb)^(k-1)*exp(-(x/lamb)^k)
*((1-exp(-(x/lamb)^k))^(alpha-1))
}

# weibull Exp- Cumulative distribution function.
cdf_Wexp <- function(par,x){
alpha  = par[1]
k      = par[2]
lamb   = par[3]
(1-exp(-(x/lamb)^k))^alpha
}
goodness.fit(pdf=pdf_Wexp, cdf=cdf_Wexp,
starts = c(1, 1, 1), data = x, method="N",
domain=c(0,1),mle=NULL)

#----------------------------------------------------
# Beta Exp- pdf distribution function.

pdf_Bexp <- function(par,x){
a      = par[1]
b      = par[2]
lamb   = par[3]
(lamb/beta(a,b))*exp(-lamb*x)*(1-exp(-lamb*x))
^(a-1)*(exp(-lamb*x))^(b-1)
}

# Beta Exp- Cumulative distribution function.
cdf_Bexp <- function(par,x){
a      = par[1]
```

```
b      = par[2]
lamb   = par[3]
pbeta((1-exp(-lamb*x)),a,b)
}
goodness.fit(pdf=pdf_Bexp, cdf=cdf_Bexp,
 starts = c(1, 1, 1),
 data = x, method="S", domain=c(0,1),
mle=NULL)


#----------------------------------------------

# Weibull - pdf distribution function.

pdf_W <- function(par,x){
alpha      = par[1]
lambda     = par[2]
((alpha/lambda)*(x/lambda)^(alpha-1))*
exp(-(x/lambda)^alpha)
}

# Weibull- Cumulative distribution function.
cdf_W <- function(par,x){
alpha      = par[1]
lambda     = par[2]
(1-exp(-(x/lambda)^alpha))
}
goodness.fit(pdf=pdf_W, cdf=cdf_W,
starts = c(.1, 1),
data = x, method="N", domain=c(0,1),
mle=NULL)
```

```
#                    SECANT WEIBULL

#Km-Weibull  Probability density function.
pdf_KumW <- function(par,x){
a= par[1]
b= par[2]
c= par[3]
lambda= par[4]
#beta=par[5]
((pi/3))*(a*b*c*lambda^(c)*x^(c-1)*exp(-(lambda*x)^c)
*(1-exp(-(lambda*x)^c))^(a-1)*(1-(1-exp(-(lambda*x)^c))
^a)^(b-1))*sec((pi/3)*((1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
*tan((pi/3)*((1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
}

# km Weibull - Cumulative distribution function.
cdf_KumW <- function(par,x){
if(x>=.Machine$double.xmax) return(1)
a= par[1]
b= par[2]
c= par[3]
lambda= par[4]
#beta=par[5]
(sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b)))-1
}

aic=352
aInicial       = .1
bInicial       = .1
cInicial       = .1
lambdaInicial  = .1
```

```
a = 20
#aic=386.1
#####  USAR O A FUNCAO Nelder-Mead
# valor que o laço deve parar caso encontre um a menor ou igual a este
criterio    = 351
incremento  =  .1
#aqui abaixo fica o valor maximo que o laço deve percorrer
max = 8
#variaveis que identificam o menor a e seus respectivos i j k y
minA = 20
minAIC=352
minI = 0
minJ = 0
minK = 0
minY = 0

i = aInicial
while(i <= max && abs(aic) > criterio ){
  j = bInicial
  while(j <= max && abs(aic) > criterio){
    k = cInicial
    while(k <= max && abs(aic) > criterio){
      #w =  betaInicial
        # while(w <= max && a > criterio){
            y =lambdaInicial
               while(y <= max && abs(aic) > criterio){
      tryCatch(
{
  ajuste<-goodness.fit(pdf=pdf_KumW, cdf=cdf_KumW,
  starts = c(i, j, k, y), data = x, method="N",
  domain=c(0,.Machine$double.xmax),mle=NULL)
  a = ajuste$A
```

```
  aic = abs(ajuste$AIC)#COLOQUEI O ABS
  a = goodness.fit(pdf=pdf_KumW, cdf=cdf_KumW,
starts = c(i, j, k, y), data = x, method="N",
  domain=c(0,Inf),mle=NULL)$A
  aic = goodness.fit(pdf=pdf_KumW, cdf=cdf_KumW,
starts = c(i, j, k, y), data = x, method="N",
domain=c(0,Inf),mle=NULL)$AIC
},
error = function(cond) {
  a = 20
  return(a)
})
if(is.nan(a)){
  a = 20
  aic=351
}
#if(aic<0){
 # a = 20
#  aic=386.1
#}

#if(a < minA){
if(aic < minAIC){
  minA = a
  minAIC=abs(aic)
  minI = i
  minJ = j
  minK = k
  minY = y
}
cat("i = ", i, "j = ", j, "k = ", k,
"y = ", y, "aic = ", aic, "a = ", a, "\n")
```

```
y = y + incremento
    }
k = k + incremento
    }
j = j + incremento
  }
i = i + incremento
}

SAIDA=c(minAIC,minI,minJ,minK,minY)
SAIDA
goodness.fit(pdf=pdf_KumW, cdf=cdf_KumW,
starts = c(minI,minJ,minK,minY), data = x,
 method="N", domain=c(0,Inf),mle=NULL)$AIC
MLE=goodness.fit(pdf=pdf_KumW, cdf=cdf_KumW,
starts = c(minI,minJ,minK,minY), data = x,
method="N", domain=c(0,Inf),mle=NULL)$mle

MLE=cbind(MLE)
mleI=MLE[1,]
mleJ=MLE[2,]
mleK=MLE[3,]
mleY=MLE[4,]


DsKumW=function(x,a,b,c,lambda){
((pi/3))*(a*b*c*lambda^(c)*x^(c-1)*
exp(-(lambda*x)^c)*(1-exp(-(lambda*x)^c))
^(a-1)*(1-(1-exp(-(lambda*x)^c))^a)^(b-1))
*sec((pi/3)*((1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
*tan((pi/3)*((1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
}
```

```
KumWP=function(x,a,b,c,lambda,beta){
(a*b*c*lambda*(beta^c)*(x^(c-1))*((1-exp(-(x*beta)^c))
^(a-1))*((1-(1-exp(-(beta*x)^c))^a)^(b-1))*
exp(-lambda*(1-(1-(1-exp(-(beta*x)^c))^a)^b) -
 (beta*x)^c))/(1-exp(-lambda))
}

#----------------------------------------------------
#Km-Weibull  Probability density function.
pdf_KumW <- function(par,x){
a= par[1]
b= par[2]
c= par[3]
lambda= par[4]
#beta=par[5]
((pi/3))*(a*b*c*lambda^(c)*x^(c-1)*exp(-(lambda*x)^c)
*(1-exp(-(lambda*x)^c))^(a-1)*(1-(1-exp(-(lambda*x)^c))
^a)^(b-1))*sec((pi/3)*((1-(1-(1-exp(-(lambda*x)^c))^a)
^b)))*tan((pi/3)*((1-(1-(1-exp(-(lambda*x)^c))^a)^b)))
}

# km Weibull - Cumulative distribution function.
cdf_KumW <- function(par,x){
if(x>=.Machine$double.xmax) return(1)
a= par[1]
b= par[2]
c= par[3]
lambda= par[4]
#beta=par[5]
(sec((pi/3)*(1-(1-(1-exp(-(lambda*x)^c))^a)^b)))-1
}
```

```
goodness.fit(pdf=pdf_KumW,
cdf=cdf_KumW,starts = c(.01,.1,.1,.01),
data = x, method="N",   domain=c(0,Inf),mle=NULL)

#----------------------------------------------------------------------



#par(mfrow = c(1,1)), pty = "s")
#par(pin = c(5.5, 5.5,5.5, 5.5))
#par(mar=c(2,2,2,2))
#win.graph(width=12, height=12)
hist(x,probability=T, col="white",ylim=c(0,0.03),
xlab="x",main="",ylab="pdf")
#1.07261933, 0.08285035, 0.71555724, 0.48873123
#mleI,mleJ,mleK,mleY
#0.027744077, 0.432973830, 5.706656211, 0.004070418
curve(DsKumW(x,0.01449035, 0.39489463, 5.61304343, 0.01540161
),lty = 1,
lwd=2,col="blue",add=T)
curve(KumWP(x,  13.68418891,  1.34300734,  1.54411191,
2.05262886, 0.01718915),lty = 1,lwd=2,col="red",add=T)
curve(KumW(x,  1.840154201, 0.584271412, 3.655399606,
0.008861448),lty = 1,lwd=2,col="green",add=T)
curve(Wpdf(x,  3.898199146, 0.007218809),lty = 1,lwd=2,
col="black",add=T)
legend('top', c(expression(
list(Sec-KumW),
list(Kum-WP),
list(Kum-W),
list(W)
)),ncol=1,bty="n",col=c("blue","red","green","black"),
lty=1,cex =0.8,lwd=2)
```

```
plot.ecdf(x, col="violet",ylim=c(0,1),xlab="x",
main="",ylab="cdf")
curve(SecKumWcdf(x,4.78426935,  1.99058458,
0.96324017, 20.38750830),lty = 1,lwd=2,
col="blue",add=T)
curve(KumWPcdf(x,  13.68418891,  1.34300734,
1.54411191, -2.05262886,  0.01718915),lty = 1,
 lwd=2,col="red",add=T)
curve(KumWcdf(x,  1.840154201, 0.584271412,
3.655399606, 0.008861448),lty = 1,lwd=2,
col="green",add=T)
curve(Wcdf(x,  3.898199146, 0.007218809),
lty = 1,lwd=2,col="black",add=T)
legend('bottom', c(expression(
list(Sec-KumW),
list(Kum-WP),
list(Kum-W),
list(W)
)),ncol=1,bty="n",col=c("blue","red","green",
"black"),lty=1,cex =0.8,lwd=2)



###########################################################################
#                    UNIVERSIDADE FEDERAL RURAL DE PERNAMBUCO
#           PROGAMA DE POS-GRADUACAO EM BIOMETRIA E ESTATISTICA APLICADA
#                     CODIGO FONTE DA SIMULAÇÃO NUMÉRICA
#                               BY LUCIANO SOUZA
###########################################################################

require(pracma)
```

```
require(AdequacyModel)
#' esta funcao retorna uma amostra baseado nos slides de mcmc
#' @param n_sample o tamanho da amostra
#' @param lambda o parametro
#' @param density a funcao de densidade com o corpo (x, lambda)
#' @param limsup limite superior

#get_sample <- function(n_sample, lambda, density, limsup){



#---------------------colocar aqui o novo get_sample----------------------------------
#M=1
get_sample <- function(n_sample, lambda, density, limsup){
sample<-rep(NaN,n_sample)
cont<-0
for(i in 1:n_sample) {
x<-runif(1,max=limsup+1)
u<-runif(1)
while(u>density(x, lambda)) {
x<-runif(1,max=limsup+1)
u<-runif(1)
cont<-cont+1
}
sample[i]<-x
}
cat("\ntaxa de aceitação: ",n_sample/cont)
return(sample)
}

#--------------------------------------------------------------------------------------
```

```
#Retirei o codigo *

#' funcao que retorna um vetor com a media, variancia, eqm e bias
#' @param lambda parametro que se deseja estimar
#' @param random funcao aleatorio
#' @param density1 funcao de densidade com escopo (x, lambda)
#' @param density2 funcao de densidade com escopo (par, x)
#' @param cumulate funcao acumulada com escopo (par, x)
#' @param n_replica numero de replicas
#' @param n_sample tamanho da amostra
#' @return retorna um vetor com lambda, variancia, media, bias e eqm
calculate = function(lambda, random, density1, density2, cumulate,
                     n_replica, n_sample){
  result_sample = c()
  Dn = c()
  EQM = c()
  variance = c()
  limsup = random(0.999999999,lambda)
  for(i in 1:n_replica){

    sample = get_sample(n_sample, lambda, density1, limsup)

    lambda_estimado = goodness.fit(pdf=density2, cdf=cumulate, starts = c(1),
                                   data = sample, method="N",
                                   domain=c(0,Inf),mle=NULL)$mle

    result_sample[i] = lambda_estimado
     s2<-sample
empirical_CDF<-function(y){
retval<-rep(0,length(y))
for(i in 1:length(y)) {
retval[i]<-length(s2[s2<y[i]])/length(s2)
```

```
}
return(retval)
}

# Dn[i]<-max(abs(SinExp_CDF(sample, lambda_estimado)-empirical_CDF(sample)))
# Dn[i]<-max(abs(SinExp_CDF(sample, lambda_estimado)-SinExp_CDF(sample, lambda)))
    #The K-S test is based on the maximum distance between these two curves.
      Dn[i]<-ks.test(SinExp_CDF(sample, lambda_estimado),
        SinExp_CDF(sample, lambda))$statistic
  variance[i]<-J(lambda_estimado,sample)
  }
  ERROR=mean(Dn)
  media = mean(result_sample)
  variance<-mean(variance)
  bias = mean(result_sample) - lambda
  EQM = variance+bias^2
  return(c(lambda, variance, media, bias, EQM, ERROR))
}


#---------------EXECUCAO------SIN -----------------------------
# funcao density1, utilizada no get_sample
SinExp = function(x,lambda){
  (lambda*pi/2)*exp(-(lambda*x))*sin((pi/2)*exp(-(lambda*x)))
}

# funcao acumulada
SinExp_CDF = function(x,lambda){
  cos((pi/2)*exp(-(lambda*x)))
}

# funcao random, utilizada para o limsup
```

```
# basta passar ela por parametro
rSinexp = function(U,lambda){
  (-1*log(acos(U)*2/pi)/lambda)
}



#utilizada no Adequacy
# pdf distribution function.
pdf_exp <- function(par,x){
  lambda     = par[1]
(lambda*pi/2)*exp(-(lambda*x))*sin((pi/2)*exp(-(lambda*x)))
}

#utilizada no Adequacy
# cdf distribution function
cdf_exp <- function(par,x){
  lambda     = par[1]
   cos((pi/2)*exp(-(lambda*x)))
 }




#-------------------FIM EXECUCAO SIN -----------------------------
#Fisher

J<-function(lambda, x){
soma<-0
for (i in 1:length(x)) {
soma=+4*cos((1/2)*pi*(-1+exp(-lambda*x[i])))^2*lambda^2/(2*cos((1/2)*pi*
(-1+exp(-lambda*x[i])))*sin((1/2)*pi*(-1+exp(-lambda*x[i])))*pi
*exp(-lambda*x[i])*lambda^2*x[i]^2+pi^2
```

```
*exp(-2*lambda*x[i])*lambda^2*x[i]^2+4*cos((1/2)*pi*(-1+exp(-lambda*x[i])))^2)
}
return(soma)
}


#criando matriz e salvando em arquivo
amostra = c(50,100,200,1000)
nreplicas = 5000
lambdas = c(0.5,1,1.5,2)
resultado_final = matrix(nrow=length(lambdas), ncol=7)
colnames(resultado_final) = list('Amostra', 'Lambda', 'Media',
                                 'Variancia', 'Bias', 'EQM','ERROR')
result<-NULL
for(k in 1:length(amostra)){
   for(i in 1:length(lambdas)){
    result = rbind(result, calculate(lambda = lambdas[i], random = rSinexp,
                       density1 = SinExp, density2 =pdf_exp,
                       cumulate = cdf_exp,
                       n_replica = nreplicas, n_sample = amostra[k]))
  resultado_final[i,1] = amostra[k]
  resultado_final[i,2] = lambdas[i]
  resultado_final[i,3] = round(result[3],6)
  resultado_final[i,4] = round(result[2],6)
  resultado_final[i,5] = abs(round(result[4],6))
  resultado_final[i,6] = round(result[5],6)
  resultado_final[i,7] = round(result[6],6)
}
}
write.table(x = resultado_final, sep = ';',
  file = 'C:/Users/IOLANDA FAMILIA/Dropbox/TESE/SinExp/teste_50_100_200_1000.txt')
```

```
#' esta funcao retorna uma amostra baseado nos slides de mcmc
#' @param n_sample o tamanho da amostra
#' @param lambda o parametro
#' @param density a funcao de densidade com o corpo (x, lambda)
#' @param limsup limite superior

#get_sample <- function(n_sample, lambda, density, limsup){


#---------------------colocar aqui o novo get_sample------------------------------------
#M=1
get_sample <- function(n_sample, lambda, density, limsup){
sample<-rep(NaN,n_sample)
cont<-0
for(i in 1:n_sample) {
x<-runif(1,max=limsup+1)
u<-runif(1)
while(u>density(x, lambda)) {
x<-runif(1,max=limsup+1)
u<-runif(1)
cont<-cont+1
}
sample[i]<-x
}
cat("\ntaxa de aceitação: ",n_sample/cont)
return(sample)
}

#--------------------------------------------------------------------------------------------
```

```
#Retirei o codigo *

#' funcao que retorna um vetor com a media, variancia, eqm e bias
#' @param lambda parametro que se deseja estimar
#' @param random funcao aleatorio
#' @param density1 funcao de densidade com escopo (x, lambda)
#' @param density2 funcao de densidade com escopo (par, x)
#' @param cumulate funcao acumulada com escopo (par, x)
#' @param n_replica numero de replicas
#' @param n_sample tamanho da amostra
#' @return retorna um vetor com lambda, variancia, media, bias e eqm
calculate = function(lambda, random, density1, density2, cumulate,
                     n_replica, n_sample){
  result_sample = c()
  Dn = c()
  EQM = c()
  variance = c()
  limsup = random(0.999999999,lambda)
  for(i in 1:n_replica){

    sample = get_sample(n_sample, lambda, density1, limsup)

    lambda_estimado = goodness.fit(pdf=density2, cdf=cumulate, starts = c(1),
                               data = sample, method="N",
                               domain=c(0,Inf),mle=NULL)$mle

    result_sample[i] = lambda_estimado
     s2<-sample
empirical_CDF<-function(y){
retval<-rep(0,length(y))
for(i in 1:length(y)) {
```

```
retval[i]<-length(s2[s2<y[i]])/length(s2)
}
return(retval)
}

# Dn[i]<-max(abs(CosExp_CDF(sample, lambda_estimado)-
empirical_CDF(sample)))
# Dn[i]<-max(abs(CosExp_CDF(sample, lambda_estimado)-
CosExp_CDF(sample, lambda)))
      Dn[i]<-ks.test(CosExp_CDF(sample, lambda_estimado),
      CosExp_CDF(sample, lambda))
      $statistic
variance[i]<-J(lambda_estimado,sample)
  }
  ERROR=mean(Dn)
  media = mean(result_sample)
  variance<-mean(variance)
  bias = mean(result_sample) - lambda
  EQM = variance+bias^2
  return(c(lambda, variance, media, bias, EQM, ERROR))
}


#----------------------EXECUCAO------COS ---------------------------


CosExp = function(x,lambda){
  (lambda*pi/2)*exp(-(lambda*x))*cos((pi/2)*exp(-(lambda*x)))
}

# funcao acumulada
CosExp_CDF = function(x,lambda){
```

```
  1-sin((pi/2)*exp(-(lambda*x)))
}

# funcao random, utilizada para o limsup
# basta passar ela por parametro
rCosExp = function(U,lambda){
  (-1*log(asin(1-U)*2/pi)/lambda)
}



#utilizada no Adequacy
# pdf distribution function.
pdf_exp <- function(par,x){
  lambda     = par[1]
(lambda*pi/2)*exp(-(lambda*x))*cos((pi/2)*exp(-(lambda*x)))
}

#utilizada no Adequacy
# cdf distribution function
cdf_exp <- function(par,x){
  lambda     = par[1]
   1-sin((pi/2)*exp(-(lambda*x)))
}

#------------------FIM---EXECUCAO------COS ---------------------------
#Fisher

J<-function(lambda, x){
soma<-0
for (i in 1:length(x)) {
soma=+4*cos((1/2)*pi*exp(-lambda*x[i]))^2*lambda^2
/(2*cos((1/2)*pi*exp(-lambda*x[i]))
```

```
*sin((1/2)*pi*exp(-lambda*x[i]))*pi*exp(-lambda*x[i])*lambda^2*
x[i]^2+pi^2*exp(-2*lambda*x[i])*lambda^2
*x[i]^2+4*cos((1/2)*pi*exp(-lambda*x[i]))^2)
}
return(soma)
}


#criando matriz e salvando em arquivo
amostra = c(50,100,200,1000)
nreplicas = 5000
lambdas = c(0.5,1,1.5,2)
resultado_final = matrix(nrow=length(lambdas), ncol=7)
colnames(resultado_final) = list('Amostra', 'Lambda', 'Media',
                                 'Variancia', 'Bias', 'EQM','ERROR')
result<-NULL
for(k in 1:length(amostra)){
   for(i in 1:length(lambdas)){
    result = rbind(result, calculate(lambda = lambdas[i], random = rCosExp,
                       density1 = CosExp, density2 =pdf_exp,
                       cumulate = cdf_exp,
                       n_replica = nreplicas, n_sample = amostra[k]))
  resultado_final[i,1] = amostra[k]
  resultado_final[i,2] = lambdas[i]
  resultado_final[i,3] = round(result[3],6)
  resultado_final[i,4] = round(result[2],6)
  resultado_final[i,5] = abs(round(result[4],6))
  resultado_final[i,6] = round(result[5],6)
  resultado_final[i,7] = round(result[6],6)
}
}
write.table(x = resultado_final, sep = ';',
```

```
 file = 'C:/Users/IOLANDA FAMILIA/Dropbox/TESE/CosExp/teste_50_100_200_1000.txt')
```

```
require(pracma)
require(AdequacyModel)
#' esta funcao retorna uma amostra baseado nos slides de mcmc
#' @param n_sample o tamanho da amostra
#' @param lambda o parametro
#' @param density a funcao de densidade com o corpo (x, lambda)
#' @param limsup limite superior

#get_sample <- function(n_sample, lambda, density, limsup){


#---------------------colocar aqui o novo get_sample-----------------------------------
#M=1
get_sample <- function(n_sample, lambda, density, limsup){
sample<-rep(NaN,n_sample)
cont<-0
for(i in 1:n_sample) {
x<-runif(1,max=limsup+1)
u<-runif(1)
while(u>density(x, lambda)) {
x<-runif(1,max=limsup+1)
u<-runif(1)
cont<-cont+1
}
```

```
sample[i]<-x
}
cat("\ntaxa de aceitação: ",n_sample/cont)
return(sample)
}

#----------------------------------------------------------------------------------------------------


#Retirei o codigo *

#' funcao que retorna um vetor com a media, variancia, eqm e bias
#' @param lambda parametro que se deseja estimar
#' @param random funcao aleatorio
#' @param density1 funcao de densidade com escopo (x, lambda)
#' @param density2 funcao de densidade com escopo (par, x)
#' @param cumulate funcao acumulada com escopo (par, x)
#' @param n_replica numero de replicas
#' @param n_sample tamanho da amostra
#' @return retorna um vetor com lambda, variancia, media, bias e eqm
calculate = function(lambda, random, density1, density2, cumulate,
                     n_replica, n_sample){
  result_sample = c()
  Dn = c()
  EQM = c()
  variance = c()
  limsup = random(0.999999999,lambda)
  for(i in 1:n_replica){

    sample = get_sample(n_sample, lambda, density1, limsup)

    lambda_estimado = goodness.fit(pdf=density2, cdf=cumulate, starts = c(1),
```

```
                                    data = sample, method="N",
                                    domain=c(0,Inf),mle=NULL)$mle

    result_sample[i] = lambda_estimado
     s2<-sample
empirical_CDF<-function(y){
retval<-rep(0,length(y))
for(i in 1:length(y)) {
retval[i]<-length(s2[s2<y[i]])/length(s2)
}
return(retval)
}

# Dn[i]<-max(abs(TanExp_CDF(sample, lambda_estimado)
-empirical_CDF(sample)))
# Dn[i]<-max(abs(TanExp_CDF(sample, lambda_estimado)
-TanExp_CDF(sample, lambda)))
      Dn[i]<-ks.test(TanExp_CDF(sample, lambda_estimado),
      TanExp_CDF(sample, lambda))$
      statistic
variance[i]<-J(lambda_estimado,sample)
  }
  ERROR=mean(Dn)
  media = mean(result_sample)
  variance<-mean(variance)
  bias = mean(result_sample) - lambda
  EQM = variance+bias^2
  return(c(lambda, variance, media, bias, EQM, ERROR))
}


#---------------EXECUCAO---------------------------------------
```

```
# funcao density1, utilizada no get_sample
TanExp = function(x,lambda){
  (lambda*pi/4)*exp(-(lambda*x))*(sec((pi/4)-(pi/4)*exp(-(lambda*x))))^2
}

# funcao acumulada
TanExp_CDF = function(x,lambda){
  cot((pi/4)+(pi/4)*exp(-(lambda*x)))
}

# funcao random, utilizada para o limsup
# basta passar ela por parametro
rTanexp = function(U,lambda){
  (-1*log(acot(U)*4/pi-1)/lambda)
}

#Fisher

J<-function(lambda, x){
soma<-0
for (i in 1:length(x)) {
soma=+8*lambda^2*(cos((1/4)*pi*(1+exp(-lambda*x[i])))^2-1)
/(-4*cos((1/4)*pi*(1+exp(-lambda*x[i])))
*pi*exp(-lambda*x[i])*lambda^2*x[i]^2*sin((1/4)*pi*
(1+exp(-lambda*x[i])))+pi^2*exp(-2*lambda*x[i])*lambda^2*x[i]^2+
8*cos((1/4)*pi*(1+exp(-lambda*x[i])))^2-8)
}
return(soma)
}

#utilizada no Adequacy
# pdf distribution function.
```

```
pdf_exp <- function(par,x){
  lambda     = par[1]
(lambda*pi/4)*exp(-(lambda*x))*(sec((pi/4)-(pi/4)*
exp(-(lambda*x))))^2
}

#utilizada no Adequacy
# cdf distribution function
cdf_exp <- function(par,x){
  lambda     = par[1]
   #tan((pi/4)-(pi/4)*exp(-(lambda*x)))
   cot((pi/4)+(pi/4)*exp(-(lambda*x)))
}

#criando matriz e salvando em arquivo
amostra = c(50,100,200,1000)
nreplicas = 5000
lambdas = c(0.5,1,1.5,2)
resultado_final = matrix(nrow=length(lambdas), ncol=7)
colnames(resultado_final) = list('Amostra', 'Lambda', 'Media',
                                 'Variancia', 'Bias', 'EQM','ERROR')

result<-NULL
for(k in 1:length(amostra)){
  for(i in 1:length(lambdas)){
        result = rbind(result, calculate(lambda = lambdas[i], random = rTanexp,
                       density1 = TanExp, density2 =pdf_exp,
                       cumulate = cdf_exp,
                       n_replica = nreplicas, n_sample = amostra[k]))
  resultado_final[i,1] = amostra[k]
  resultado_final[i,2] = lambdas[i]
  resultado_final[i,3] = round(result[3],6)
```

```
  resultado_final[i,4] = round(result[2],6)
  resultado_final[i,5] = abs(round(result[4],6))
  resultado_final[i,6] = round(result[5],6)
  resultado_final[i,7] = round(result[6],6)
}
write.table(x = resultado_final, sep = ';',
 file = 'C:/Users/IOLANDA FAMILIA/Dropbox/TESE/TanExp/teste_50_100_200_1000.txt')



#------------------------INICIO DA SIMULACAO SEC AMOSTRA 100-------------------------

require(pracma)
require(AdequacyModel)
#' esta funcao retorna uma amostra baseado nos slides de mcmc
#' @param n_sample o tamanho da amostra
#' @param lambda o parametro
#' @param density a funcao de densidade com o corpo (x, lambda)
#' @param limsup limite superior

#get_sample <- function(n_sample, lambda, density, limsup){


#---------------------colocar aqui o novo get_sample-----------------------------------
#M=1
get_sample <- function(n_sample, lambda, density, limsup){
sample<-rep(NaN,n_sample)
cont<-0
for(i in 1:n_sample) {
x<-runif(1,max=limsup+1)
u<-runif(1)
while(u>density(x, lambda)) {
```

```
x<-runif(1,max=limsup+1)
u<-runif(1)
cont<-cont+1
}
sample[i]<-x
}
cat("\ntaxa de aceitação: ",n_sample/cont)
return(sample)
}

#------------------------------------------------------------------------------------------------------------


#Retirei o codigo *

#' funcao que retorna um vetor com a media, variancia, eqm e bias
#' @param lambda parametro que se deseja estimar
#' @param random funcao aleatorio
#' @param density1 funcao de densidade com escopo (x, lambda)
#' @param density2 funcao de densidade com escopo (par, x)
#' @param cumulate funcao acumulada com escopo (par, x)
#' @param n_replica numero de replicas
#' @param n_sample tamanho da amostra
#' @return retorna um vetor com lambda, variancia, media, bias e eqm
calculate = function(lambda, random, density1, density2, cumulate,
                     n_replica, n_sample){
  result_sample = c()
  Dn = c()
  EQM = c()
  variance = c()
  limsup = random(0.999999999,lambda)
  for(i in 1:n_replica){
```

```
    sample = get_sample(n_sample, lambda, density1, limsup)

    lambda_estimado = goodness.fit(pdf=density2, cdf=cumulate, starts = c(1),
                                   data = sample, method="N",
                                   domain=c(0,Inf),mle=NULL)$mle

    result_sample[i] = lambda_estimado
     s2<-sample
empirical_CDF<-function(y){
retval<-rep(0,length(y))
for(i in 1:length(y)) {
retval[i]<-length(s2[s2<y[i]])/length(s2)
}
return(retval)
}

# Dn[i]<-max(abs(SecExp_CDF(sample, lambda_estimado)-
empirical_CDF(sample)))
# Dn[i]<-max(abs(SecExp_CDF(sample, lambda_estimado)-
SecExp_CDF(sample, lambda)))
      Dn[i]<-ks.test(SecExp_CDF(sample, lambda_estimado),
      SecExp_CDF(sample, lambda))$
      statistic
variance[i]<-J(lambda_estimado,sample)
  }
  ERROR=mean(Dn)
  media = mean(result_sample)
  variance<-mean(variance)
  bias = mean(result_sample) - lambda
  EQM = variance+bias^2
  return(c(lambda, variance, media, bias, EQM, ERROR))
```

```
}



#---------------EXECUCAO------SEC -----------------------------
# funcao density1, utilizada no get_sample
SecExp = function(x,lambda){
  (1/3)*csc((1/6)*pi+(1/3)*pi*exp(-lambda*x))*cot((1/6)*pi+(1/3)*pi
  *exp(-lambda*x))*pi*lambda*exp(-lambda*x)
}

# funcao acumulada
SecExp_CDF = function(x,lambda){
 csc((1/6)*pi+(1/3)*pi*exp(-lambda*x))-1
}

# funcao random, utilizada para o limsup
# basta passar ela por parametro
rSecExp = function(U,lambda){
  (-1*log(3/pi*acsc(U+1)-1/2)/lambda)
}



#utilizada no Adequacy
# pdf distribution function.
pdf_exp <- function(par,x){
  lambda     = par[1]
(1/3)*csc((1/6)*pi+(1/3)*pi*exp(-lambda*x))*cot((1/6)*pi+(1/3)*pi
*exp(-lambda*x))*pi*lambda*exp(-lambda*x)
}

#utilizada no Adequacy
# cdf distribution function
```

```
cdf_exp <- function(par,x){
  lambda     = par[1]
   csc((1/6)*pi+(1/3)*pi*exp(-lambda*x))-1
}

#--------------------FIM EXECUCAO SEC -----------------------------

#Fisher

J<-function(lambda, x){
soma<-0
for (i in 1:length(x)) {
soma=+9*cos((1/6)*pi*(1+2*exp(-lambda*x[i])))^2*(cos((1/6)*pi
*(1+2*exp(-lambda*x[i])))^2-1)*
lambda^2/(-3*cos((1/6)*pi*(1+2*exp(-lambda*x[i])))^3*
exp(-lambda*x[i])*pi*lambda^2*x[i]^2*sin((1/6)*pi
*(1+2*exp(-lambda*x[i])))+3*cos((1/6)*pi*(1+2*exp(-lambda*x[i])))^2*
exp(-2*lambda*x[i])*pi^2*lambda^2*x[i]^2-3
*cos((1/6)*pi*(1+2*exp(-lambda*x[i])))*exp(-lambda*x[i])*pi
*lambda^2*x[i]^2*sin((1/6)*pi
*(1+2*exp(-lambda*x[i])))-exp(-2*lambda*x[i])*pi^2*lambda^2*
x[i]^2+9*cos((1/6)*pi*(1+2*exp(-lambda*x[i])))^4-9
*cos((1/6)*pi*(1+2*exp(-lambda*x[i])))^2)

}
return(soma)
}



#criando matriz e salvando em arquivo
amostra = c(50,100,200,1000)
nreplicas = 5000
```

```
lambdas = c(0.5,1,1.5,2)
resultado_final = matrix(nrow=length(lambdas), ncol=7)
colnames(resultado_final) = list('Amostra', 'Lambda', 'Media',
                                 'Variancia', 'Bias', 'EQM','ERROR')
result<-NULL
for(k in 1:length(amostra)){
  for(i in 1:length(lambdas)){

    result = rbind(result, calculate(lambda = lambdas[i], random = rSecExp,
                       density1 = SecExp, density2 =pdf_exp,
                       cumulate = cdf_exp,
                       n_replica = nreplicas, n_sample = amostra[k]))
  resultado_final[i,1] = amostra[k]
  resultado_final[i,2] = lambdas[i]
  resultado_final[i,3] = round(result[3],6)
  resultado_final[i,4] = round(result[2],6)
  resultado_final[i,5] = abs(round(result[4],6))
  resultado_final[i,6] = round(result[5],6)
  resultado_final[i,7] = round(result[6],6)
}

write.table(x = resultado_final, sep = ';',
file = 'C:/Users/IOLANDA FAMILIA/Dropbox/TESE/SecExp/teste_50_100_200_1000.txt')
```