Ricardo José Ferreira

Development of a mixed model using Generalized Renewal Processes and the Weibull Distribution

Recife

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UNIVERSIDADE FEDERAL RURAL DE PERNAMBUCO
PRÓ-REITORIA DE PESQUISA E PÓS-GRADUAÇÃO
PROGRAMA DE PÓS-GRADUAÇÃO EM BIOMETRIA E ESTATÍSTICA APLICADA

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Tese julgada adequada para obtenção do título de Doutor em Biometria e Estatística Aplicada, defendida e aprovada por unanimidade em 29/01/2016 pela comissão examinadora

Área de concentração: Biometria e Estatística Aplicada

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Dedico esse trabalho a minha mãe. Ela me deu muita bronca pra estudar e hoje acredito esse ser mais um dos frutos que ela me incentivou a produzir. Dona Maria Salustiana, esse trabalho é seu!

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"Não vos amoldeis às estruturas deste mundo,
mas transformai-vos pela renovação da mente,
a fim de distinguir qual é a vontade de Deus:
o que é bom, o que Lhe é agradável, o que é perfeito.
(Bíblia Sagrada, Romanos 12, 2)

Abstract

In order to analyze interventions in repairable systems, the literature contains several methodologies aiming to model the behavior of times between interventions. Such interventions can be modeled by Point Stochastic Processes in order to analyze the probabilistic behavior of times between events. Specifically, the Generalized Renewal Processes allow the study of times between interventions by measuring the quality of each intervention and the response of the system to these interventions — this is done by using the concept of virtual age. In such concept it is possible to apply two kinds of Kijima models (Type I and II). Therefore, this work presents a model capable of study the quality of interventions using up of a mix between the two Kijima models where it is possible to capture the performance on each of these interventions proportionally. Specifically, a new approach to virtual age of Kijima models is presented as well as mathematical properties of the Generalized Renewal Process using the Weibull distribution probability. Finally, the applicability of the model is checked in real data from some problems found in the literature.

Key-words: Repairable Systems. Generalized Renewal Processes. Kijima models.

Resumo

Para analisar intervenções em sistemas reparáveis, a literatura apresenta diversas metodologias visando modelar o comportamento de tempos entre intervenções. Tais intervenções podem ser modeladas por Processos Estocásticos Pontuais visando analisar o comportamento probabilístico dos tempos entre eventos. Especificamente, os Processos de Renovação Generalizados permitem o estudo de tempos entre intervenções medindo a qualidade de impacto de cada intervenção e a resposta do sistema a tais intervenções — isto é feito utilizando o conceito de idade virtual. Em tal conceito é possível se aplicar dois tipos de modelos Kijima (tipo I e II).Sendo assim, esse trabalho apresenta um modelo capaz de estudar a qualidade de intervenções utilizando-se de uma mistura entre os dois modelos Kijima onde é possível capturar a atuação de cada um desses sobre as intervenções proporcionalmente. Especificamente, uma nova abordagem sobre a idade virtual dos modelos Kijima é apresentada, bem como propriedades matemáticas dos Processos de Renovação Generalizados utilizando a distribuição de probabilidade Weibull. Por fim, a aplicabilidade do modelo é verificada em dados reais de alguns problemas presentes na literatura.

Palavras-chaves: Sistemas Reparáveis. Processos de Renovação Generalizados. Modelos Kijima.

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List of abbreviations and acronyms

ABAO	As bad as old
AGAN	As good as new
BP	Brown-Proschan
CDF	Cumulative Distribution Function
GRP	Generalized Renewal Processes
KI	Kijima Type I
KII	Kijima Type II
LL	Log-Likelihood
MK	Mixed Kijima
MSE	Mean Squared Error
NHPP	Non-Homogeneous Poisson Processes
PDF	Probability Density Function
RP	Renewal Processes
WGRP	Weibull-based Generalized Renewal Processes

Contents

1 Introduction

One of the main objectives of humanity is try to extend the life time of every thing - alive or not. Here, these things are entitled of systems and a brief concept is to define a system as an object or a set of objects that performs a specific (set of) function(s). These systems can be found in several fields of knowledge - organic systems are present in Biology, Agrarian, Human Health Sciences and so on; Inorganic systems are present in Computer, Engineering, Mechanics and so forth. However, we can not limit systems only in these definitions - actually, organic systems can be though in a broader view as well as inorganic systems, but we use this concept to focus in problems that we take here. Thus, a system can be generally studied and then its characteristics can be modelled by different mathematical tools. Furthermore, these systems can be functional up to the first occurrence of an undesirable event (non-repairable systems), or they can be restored to keep functioning with certain capacity (repairable systems). This work is related to systems that can be restored by some kind of intervention - the repairable ones. This discussion can be found and is based in Rigdon & Basu (2000).

Interventions have different impacts depending on their quality and the response of the system. Dealing with that, Brown & Proschan (1983) propose a model, called as Brown-Proschan (BP) model, that tries to capture the effect of an intervention on a repairable system through a dichotomous variable, say D, where $D = 0$ means that a perfect intervention has occurred, whereas $D = 1$ means that a minimal intervention has been performed. To balance these situations, Brown & Proschan (1983) state that $P(D = 0) = p$ and $P(D = 1) = 1 - p$. Thus, they assign probabilities to these two kinds of intervention. However, this modelling is capable only to treat the occurrence of a perfect/minimal intervention with specific probabilities and does not represent other real world situations.

Trying to expand these concepts, Kijima et al. (1988) developed a new concept, namely virtual age, capable to analyze both types of intervention aforementioned and three more: imperfect, better and worse intervention. To measure these situations, a rejuvenation parameter, say q, takes values traditionally between 0 and 1. Basically, Kijima et al. (1988) present a classification where deteriorating systems can be restored to five different states, originally called as repairs.

These situations are also analyzed by Kijima (1989) and Krivtsov (2000) through

Stochastic Processes and Renewal Theory. Further, Ross (2006) cites that perfect interventions are treated through Renewal Processes (RP), whereas minimal ones are treated through Non-Homogeneous Poisson Processes (NHPP). This Stochastic Process presented by Kijima et al. (1988) is called as Generalized Renewal Processes (GRP), since it generalizes the NHPP and RP. The parameter q represents this generalization, as described above.

Kijima et al. (1988) present two kind of models that work with this parameter q. Kijima Type I tries to model situations where the intervention made acts only at the last stoppage time representing an immediate intervention. Kijima Type II tries to model situations where the intervention made acts at the whole system aiming its stoppage time history. Further, Jacopino et al. (2004) and Jacopino et al. (2006) present studies about applications of these models in real cases. They state that we could use Kijima Type I in cases where single components of a system (or a system formed by a single component), whereas Kijima Type II model is used in complex systems (with several components). However, real situations can present stoppage times that represent an intermediate situation between these two models, and this kind of situation is not treated by the most known GRP modelling literature - interventions made to impact more than the last one (Kijima I) but less than the whole history (Kijima II).

Applications of Renewal Theory are commonly found in fields of engineering where non-organic systems are analyzed and times between interventions are modelled through probabilistic distributions. Specifically, the use of GRP can be found in works as Yañez et al. (2002), Jain & Maheshwari (2006), Jimenez & Villalon (2006) where the data set is modelled via Weibull distribution. This is important, since none of these or other works present another probabilistic distribution but Weibull distribution. Furthermore, Guo et al. (2007) state the difficulty to work with GRP, since theoretical properties are not explored in literature.

This work brings a discussion about repairable systems and methods used to model time between interventions on them. Specifically, it is presented an alternative for modelling the quality and effect of interventions made on those systems, and also the impact of different types of intervention on virtual age. Furthermore, this is made by using a mixed model of Kijima approaches through GRP. Finally, using a parameter to relate times between interventions and ther types, it is introduced an idea on how to relate the types of interventions to predicted times. This work brings details of mathematical properties and results presented in Ferreira et al. (2015) where this model was firstly presented in the literature.

1.1 General Objective

The main objective of this work is to develop a mixed model based on WGRP using Kijima models. This model is capable of measuring the impact of each Kijima model in times between interventions.

1.2 Specific Objectives

To achieve the main objective, some specific objectives must be met, such as:

- To make a literature review with main works concerning GRP-WGRP modelling;
- To present advantages, properties and connective points between Kijima models;
- To develop the use of these models with the Weibull distribution - the proposed mixed model;
- To present mathematical, theoretical and pratical features of the new model;
- To apply the proposed model in real world databases.

This work is structured as follows. Chapter 2 brings a discussion about Renewal Theory. Chapter 3 presents the whole structure of the proposed model. Chapter 4 presents the main results obtained by the proposed model in cases from literature. Finally, Chapter 5 brings some discussions and conclusions about this work.

2 Literature Review

This chapter brings a review about works that use GRP framework. Several authors use this methodology to create hybrid models capable of dealing with different and specific problems. However, all problems have similar configurations - renewable systems where it is important to analyze their capacity of restoration.

2.1 Renewal Theory and the beginning of GRP

The need to deal with complex systems brought up the development of methodologies capable of analyzing behaviour of failures. The major part of developments appear to be concerned with Inorganic Systems aiming the analysis of reliability, availability and preventive maintenance policies of equipment/devices.

In this context, Crow (1975) presents a wide discussion about repairable systems and tools to evaluate their reliability. Through these tools, Point Stochastic Processes have the function to analyze time between interventions but neglecting the duration of intervention times. In other words, cases where the intervention time is considerably small compared with operational time of the system are analyzed by means of these processes.

Among the already cited processes, some of them has a wide applicability, as can be seen in Ross (2006). Cases where system returns as a new one, we have the Renewal Processes (RP). In cases where systems receive a minimal intervention, we have the Non-Homogeneous Poisson Process (NHPP).

However, these cases do not represent the reality of a number of complex systems - they suffer some kind of intervention and turn back into operation "better than old but worse than new", the so-called imperfect intervention. Some authors developed methodologies that tried to incorporate this kind of intervention to repairable systems. Brown & Proschan (1983) present the BP model which tries to incorporate the imperfect intervention using a Bernoulli variable D - the perfect intervention occurs with probability $P(D = 0) = p$ whereas the minimal one occurs with $P(D = 1) = (1 - p)$. Clearly, this modelling does not include the imperfect case since the variable is dichotomous. After this, Kijima & Sumita (1986) present a methodology capable of modelling imperfect interventions and generalize the other situations aforementioned. This modelling is known as GRP and is capable of modelling situations of RP, NHPP and imperfect interventions.

This is possible due to the concept presented in Kijima et al. (1988) known as the virtual age. This concept acts on the real age of the system and considers the impact of an intervention made after a stoppage event. Thus, the virtual age is a modification of the real age depending on the intervention made and the response of the system to it.

The virtual age is measured through a rejuvenation parameter, q, which captures the impact of interventions in operational time. Furthermore, the way that this parameter can be used in different ways depending on the Kijima model used, developed in Kijima et al. (1988).

2.1.1 The rejuvenation parameter and Kijima models

Thinking of a more general function, Baxter et al. (1982) discuss the development of a renewal function trying to generalize other processes used in this scope. Thus, seminal papers came with Kijima & Sumita (1986) and Kijima et al. (1988) where a general process is presented dealing with several situations through a new parameter, the so-called rejuvenation parameter q. It analyzes the virtual age of systems - an age different from the real one due to effects of intervetions made on the system. Traditionally, its range is between $[0, 1]$, though values less than zero or greater than one can be also considered.

As follows five situations for a deteriorating system after interventions are presented for Kijima models:

- $q < 0$: The virtual age is less than the actual age of the system. In this situation the interventions bring the system to a "better than starting" condition (KIJIMA et al., 1988).
- $q = 0$: The virtual age is reset and the system is restarted by the interventions. In other terms, the interventions lead the system to an "as good as starting" condition, reflecting a RP (MODARRES et al., 1999). The AGAN status.
- $q = 1$: The virtual age equal the actual age, *i.e.* the restoration is minimal and lead the system to an "as bad as before" intervention condition, characterizing a (NHPP) (MODARRES et al., 1999). The ABAO status.
- $0 < q < 1$: The interventions are considered imperfect, leading the system to an intermediate condition of restoration.
- $q > 1$: The interventions bring the system to a "worst than before intervention" condition. Traditionally, this situation may reflect the need of investments in the maintenance crew, once they are eventually deteriorating the system.

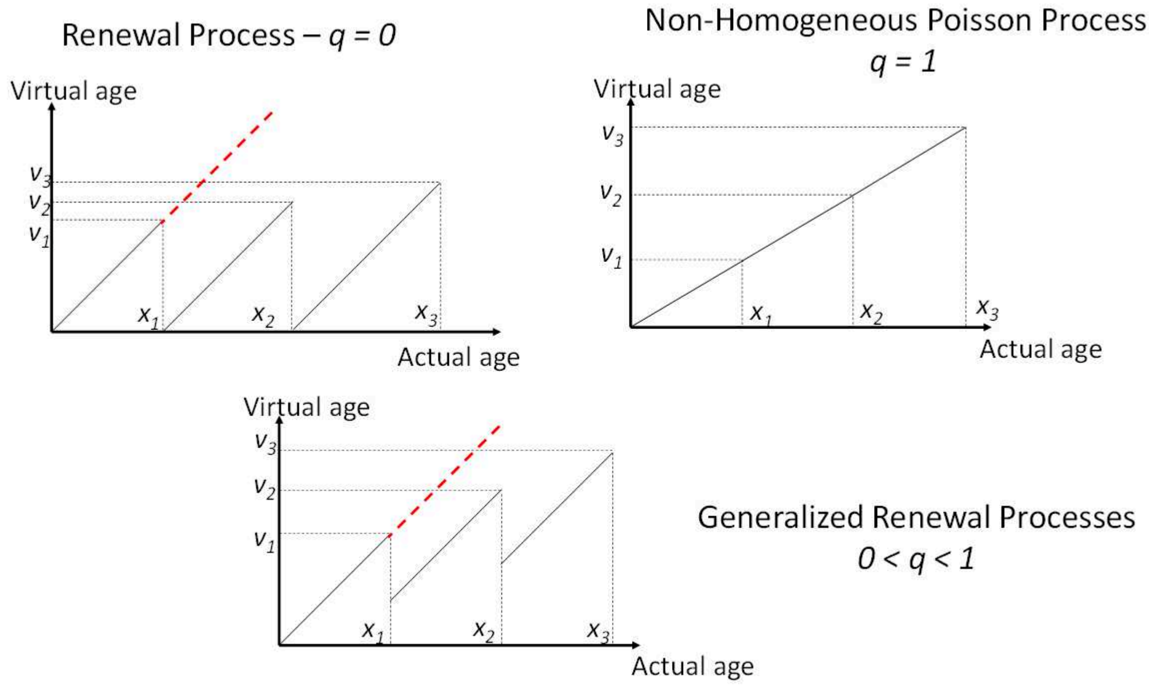

Figure 1: Three situations of interventions according with the value of q.

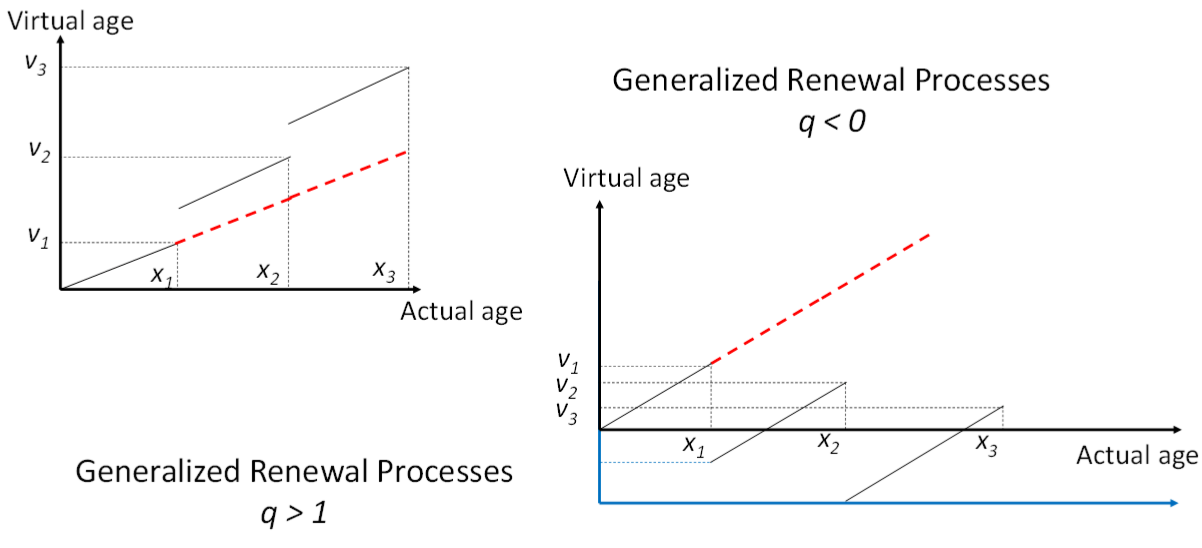

Figure 2: Two extreme situations with values of q.

We can illustrate three intermediate situations as in Figure 1 where one can see the relationship between the virtual and real ages. The real age can not be interrupted, but the virtual age suffers some kind of modification according with the value of q - the effect of the interventions. Also, the other two situations are illustrated in Figure 2.

Once explained that the rejuvenation parameter acts on the virtual age, it is important to see how this virtual age is structured mathematically. According with Kijima et al. (1988), there are two ways to capture the impact of inteventions made. They state that the intervention can act only at the last stoppage cause - it will restore the damage created by this last cause only. This situation is known as Kijima type I (KI) model.

In other way, the intervention made can have the intention to restore past problems occurred during the life time of the system. Thus, this corresponds to a deeper intervention trying to affect the whole history of the system's stoppages. This situation is known as Kijima type II (KII) model.

We can see the mathematical structure of the virtual age depending on Kijima type model as follows:

$$V_i = v(X_i \,|\, q, V_{i-1}) = V_{i-1} + qX_i \tag{2.1}$$

$$V_i = v(X_i \,|\, q, V_{i-1}) = q(V_{i-1} + X_i) \tag{2.2}$$

Where, Eq. (2.1) and Eq. (2.2) refer to KI and KII, respectively. It is easy to notice that the former brings q acting only on the last stoppage time (X_i), whereas the latter one presents the influence of q also in the system's history. With this in mind, it is important to distinguish in each problem studied and what is the impact of interventions in the repairable system.

The knowledge about this concept allows several authors to develop studies with it. These studies include properties, mathematical characteristics, applications with specific probabilistic distributions and so forth. Among them, Krivtsov (2000) presents a reviewed analysis of Generalized Renewal Processes and a discussion about the use of virtual age with the Cumulative Distribution Function (CDF). Such concepts are explored in the next section.

2.1.2 GRP functions

Let t_{i-1} be the observed cumulative time on which the $(i-1)^{th}$ intervention has in fact occurred, with respective virtual age v_{i-1}, and let $x(\geq 0)$ be the incremental time until the i^{th} intervention. Then, Kijima et al. (1988) highlight that the system has a cumulative time up to the i^{th} intervention, say T_i, which is distributed according to the following GRP cumulative distribution:

$$\begin{aligned} P(T_i \leq x + v_{i-1} | T_i > v_{i-1}) &= P(T_1 \leq x + v_{i-1} | T_1 > v_{i-1}) \\ &= \frac{F_{T_1}(x + v_{i-1}) - F_{T_1}(v_{i-1})}{1 - F_{T_1}(v_{i-1})} \end{aligned} \tag{2.3}$$

From the first equality in Equation (2.3), one can see that GRP are based on the supposition that the couple between the time until first intervention and the virtual age is sufficient to determine the family of distributions that model the times between interventions. Thus, it is supposed that the times between interventions $(X_1, X_2, \cdots)$ are identically distributed and that their eventual dependency is incorporated in the model via the virtual age function. Furthermore, the uncertainty over T_i is fully modeled once the virtual age in the $(i-1)^{th}$ intervention as well as the distribution of T_1 are known. As the time until first intervention is sometimes questionable, mainly in the cases where the involved information system has modest experience in data set maintenance or even when the system start up (T_0) is uncertain, caution must be taken in this way. A case study inspired in a data set from literature will allow to illustrate such a situation.

From Equation (2.3) and assuming its respective probability density function (PDF), $f_{T_i}(x + v_{i-1} | v_{i-1})$, one obtains the GRP hazard function:

$$h_{T_i}(x + v_{i-1} | v_{i-1}) = \frac{f_{T_1}(x + v_{i-1} | v_{i-1})}{1 - F_{T_1}(x + v_{i-1} | v_{i-1})} \tag{2.4}$$

In a general context, the hazard function can represent the instantaneous rate of interventions on the system (*e.g.* corrective or preventive actions) as time evolves. Thus, the higher h_{T_i} the lesser the time between consecutive interventions x is, reflecting system deterioration. The reasoning for systems improvement follows the same fashion. It must be highlighted that the resulting CDF and hazard functions for T_i in Equations (2.3) and (2.4) encapsulate the previous performance of the system (in terms of its technology and maintenance crew) by means of the virtual age v_{i-1}.

Once presented this structure to probabilistic distributions in GRP, it can be seen that its use is explored in Yañez et al. (2002) applying the Weibull distribution which we will call here as Weibull-Generalized Renewal Processes (WGRP). They present specific analysis considering different failure processes (time-terminated or failure-terminated), estimation processes and CDF/PDF structure.

These structures are explored and presented as follows as an overview.

2.1.3 The WGRP modeling

The most used GRP model is the WGRP, applied in some works as Yañez et al. (2002), Jain & Maheshwari (2006), and so forth. In WGRP, each time between interventions, X_i, follows a two parameters standarized Weibull distribution as shwon in Weibull (1951) conditioned on the corresponding virtual age, v_{i-1}, with shape and scale parameters β and α, in this order. Such a model is studied as follows. Actually, the interpretations presented previously for q are mostly based on WGRP. A similar representation of the cumulative distribution function (CDF) of WGRP seen in Yañez et al. (2002) is presented as follows:

$$\begin{aligned} F_{T_i}(x+v_{i-1}|v_{i-1},\alpha,\beta) &= \frac{1-\exp\left[-\left(\frac{x+v_{i-1}}{\alpha}\right)^{\beta}\right]-1+\exp\left[-\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}\right]}{1-1+\exp\left[-\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}\right]} \\ &= \frac{-\exp\left[-\left(\frac{x+v_{i-1}}{\alpha}\right)^{\beta}\right]+\exp\left[-\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}\right]}{\exp\left[-\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}\right]} \\ &= 1-\exp\left[\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}-\left(\frac{x+v_{i-1}}{\alpha}\right)^{\beta}\right] \end{aligned} \quad (2.5)$$

This distribution was structured following the conception presented in Krivtsov (2000). From the WGRP CDF, one can obtain the PDF as follows:

$$f_{T_i}(x+v_{i-1}|v_{i-1},\alpha,\beta)=\frac{\beta}{\alpha}\left(\frac{x+v_{i-1}}{\alpha}\right)^{\beta-1}\exp\left[\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}-\left(\frac{x+v_{i-1}}{\alpha}\right)^{\beta}\right] \quad (2.6)$$

This PDF is obtained through the derivative of WGRP CDF with respect to x. In turn, in the light of $F_{T_i}(x+v_{i-1}|v_{i-1},\alpha,\beta)$ and $f_{T_i}(x+v_{i-1}|v_{i-1},\alpha,\beta)$, it is straightforward to obtain the respective hazard function, following the Eq. (2.4):

$$h_{T_i}(x+v_{i-1}|v_{i-1},\alpha,\beta)=\frac{f_{T_1}(x+v_{i-1}|v_{i-1},\alpha,\beta)}{1-F_{T_1}(x+v_{i-1}|v_{i-1},\alpha,\beta)}=\frac{\beta}{\alpha}\left(\frac{x+v_{i-1}}{\alpha}\right)^{\beta-1} \quad (2.7)$$

Generally, works using these distributions are used to study life time situations involving deterioration (see Yañez et al. (2002), Veber et al. (2008), and Damaso & Garcia (2009)). Thus, both theoretical and practical results are derived from them, such as obtaining Maximum Likelihood Estimation (MLE) process, and generating sampling time through the inverse method, based on method described in Ross (2006).

Using the concept of a likelihood function, the MLE process is based on the joint PDF and derivatives, given as follows:

$$f(\mathbf{x}\,|\,\alpha,\beta,q)=\frac{\beta^n}{\alpha^{n\beta}}\left[\prod_{i=1}^{n}(x_i+v_{i-1})^{\beta-1}\right]e^{\frac{1}{\alpha^{\beta}}\left(\sum_{i=1}^{n}v_{i-1}^{\beta}-\sum_{i=1}^{n}(x_i+v_i-1)^{\beta}\right)} \quad (2.8)$$

It is worthwhile to mention that v_i encapsulates the maintenance history of the system until i^{th} intervention. Let $\ell = \ln f$ denote the log-likelihood function underlying the WGRP. Then:

$$\ell(\alpha, \beta, q|x_1, \ldots, x_n) = n(\ln\beta - \beta\ln\alpha) + (\beta - 1)\sum_{i=1}^{n}\ln(x_i + v_{i-1}) + \frac{1}{\alpha^{\beta}}\left[\sum_{i=1}^{n} v_{i-1}^{\beta} - \sum_{i=1}^{n}(x_i + v_{i-1})^{\beta}\right] \quad (2.9)$$

From ℓ, one can obtain the MLE for α by computing the α for which $\frac{\partial\ell}{\partial\alpha} = 0$:

$$\hat{\alpha} = \left[\frac{\sum_{i=1}^{n}(x_i + \hat{v}_{i-1})^{\hat{\beta}} - \sum_{i=1}^{n}\hat{v}_{i-1}^{\hat{\beta}}}{n}\right]^{\frac{1}{\hat{\beta}}}. \quad (2.10)$$

Equations (2.8), (2.9), and (2.10) show the MLE process to infer about α, β, and q and the resulting equations. However, the MLE of β and q is mathematically intricate. Equations (2.11) and (2.12) refer to the derivatives of Equation (2.9) to obtain the ML estimators for β and q. Only the derivatives are presented here once it is not possible to analytically isolate β and q as done for α. Thus, for practical purposes, $\hat{\alpha}$, $\hat{\beta}$, and $\hat{q}$ can be approximated via probabilistic optimization algorithms (*e.g.* simulated annealing, particle swarm, and genetic algorithms), where $\hat{\alpha}$ is a deterministic function of $\hat{\beta}$ and $\hat{q}$ via Equation (2.10).

$$\begin{aligned}\frac{\partial\ell}{\partial\beta} =& \frac{n}{\hat{\beta}} - n\ln\hat{\alpha} + \sum_{i=1}^{n}\ln(x_i + \hat{v}_{i-1}) \\ &- \hat{\alpha}^{-\hat{\beta}}\ln\hat{\alpha}\left[\sum_{i=1}^{n}\hat{v}_{i-1}^{\hat{\beta}} - \sum_{i=1}^{n}(x_i + \hat{v}_{i-1})^{\hat{\beta}}\right] \\ &+ \frac{1}{\hat{\alpha}^{\hat{\beta}}}\left[\sum_{i=1}^{n}\hat{v}_{i-1}^{\hat{\beta}}\ln\sum_{i=1}^{n}\hat{v}_{i-1} - \sum_{i=1}^{n}(x_i + \hat{v}_{i-1})^{\hat{\beta}}\ln\sum_{i=1}^{n}(x_i + \hat{v}_{i-1})\right] = 0\end{aligned} \quad (2.11)$$

$$\frac{\partial\ell}{\partial q} = (\hat{\beta} - 1)\sum_{i=1}^{n}\frac{\hat{v}'_{i-1}}{x_i + \hat{v}_{i-1}} + \frac{1}{\hat{\alpha}^{\hat{\beta}}}\left[\sum_{i=1}^{n}\hat{\beta}\hat{v}_{i-1}^{\hat{\beta}-1}\hat{v}'_{i-1} - \sum_{i=1}^{n}\hat{\beta}(x_i + \hat{v}_{i-1})^{\hat{\beta}-1}\hat{v}'_{i-1}\right] = 0 \quad (2.12)$$

Despite the parameter q is not explicitly showed in the derivatives, $\hat{v}$ encapsulates it by means of KI or KII.

Once presented the ML process and probabilistic functions, it is important to know how to generate random samples from this process to know its behaviour. This process is presented in the following inspection.

2.1.4 WGRP random sampling process

Sampling random WGRP series is made by means of the inverse transform method Ross (2006). Specifically, this method is based on the equality $u = 1 - R(x + v \mid v)$, where

$R(x+v \mid v) = 1 - F(x+v \mid v)$ denotes the WGRP reliability/survival function. In the inverse transform method one has that u is an instance of the random variable $U \sim Uniform(0,1)$. So, by isolating x (resorting to Eq. (2.5)) one has the WGRP instance

$$x = \alpha \left[\left(\frac{v}{\alpha} \right)^{\beta} - \ln(u) \right]^{\frac{1}{\beta}} - v. \tag{2.13}$$

Therefore, it is allowed through Eq. (2.13) to generate a sequence of n times between interventions, $(x_1, \cdots, x_i, \cdots, x_n)$, according to instances of α, β, and q. Furthermore, one can notice an important behaviour of the system according to the values of q and β in WGRP. Theoretically, from Eq. (2.7), one can see that depending on the value of β, the WGRP hazard function presents an increasing (for $\beta > 1$) or decreasing ($\beta < 1$) behaviour, since the remaining arguments of $h_{T_i}(\cdot)$ are non-negative.

This work proposes an analysis of Renewal Theory through the use of WGRP creating a mixed virtual age capable of measuring proportionally both type of Kijima models through a couple of parameters.

3 The Mixed Kijima model

In this Chapter, it is presented the proposed model, its development, properties and practical achievements. Since the virtual age is a central concept inside GRP framework, it is presented here a new approach to this concept involving a mixing of Kijima models as follows.

3.1 A New Concept of Virtual Age

Some notations are considered, focused on the problem of fitting WGRP models to time series performance data sets involving the occurrence of events of interest in a given system. Specifically, the *events of interest* will be considered as intended or unintended *interventions* on the condition of the system, and the focus will be on modeling the response of the system to these interventions in terms of the times between next interventions. Each intervention might be demanded by a single event from a set of possible (and eventually competing) ones (*e.g.* preventive and corrective actions) in such a way that the considered GRP model will incorporate the nature of such interventions in the model.

Without loss of generality, the word *time* will represent any unit measure over which the interventions are observed (*e.g.* meters, seconds, kilograms, cubic meters, and so on). Besides, the duration of each intervention is considered negligible, *i.e.* just point process are taken into account (ROSS, 1997). Finally, it is also considered that systematic increasing (decreasing) times between interventions characterize improvement (deterioration) of the system.

Definition 3.1.1. Let T_i be the time when the i^{th} intervention occurs (the actual cumulative time until i^{th} intervention) and let X_i be the time between the $(i-1)^{th}$ and the i^{th} interventions (X_0 is a non-negative constant).

From both Definition 3.1.1 and point processes foundations, we can see that $T_i = \sum_{j=1}^{i} X_j$ is the "real" age of the system when the ith intervention occurs. A direct consequence is that $T_0 = X_0$. It has been usual to assume $X_0 = 0$ in practice. It must also be highlighted that X_i (and therefore T_i) can be characterized as random variables and thus subject to statistical modeling via GRP, once they can depend on the stochastic nature of the system condition. In turn, it is also reasonable to interpret the random vectors $\mathbf{T} = (T_1, T_2, ..., T_n)$ and $\mathbf{X} = (X_1, X_2, ..., X_n)$ as stochastic processes. With these definitions in mind, we can define the new virtual age as follows.

Definition 3.1.2. Let V_i be the virtual age of the system reflecting its restoration after i interventions. Thus V_i is a function of times between interventions $\{X_j\}_{j=1}^{i}$, of the respective intervention types $\{Y_j\}_{j=1}^{i}$, and of an appending parameter, say q, $V_i = v((X_1, Y_1), (X_2, Y_2), ..., (X_i, Y_i) \,|\, q)$. Thus, this concept is expanded as a mixing of Kijima virtual age models as follows:

$$V_i = v(X_i, Y_i \,|\, q, V_{i-1}) = \theta_{Y_i}(V_{i-1} + qX_i) + (1 - \theta_{Y_i})q(V_{i-1} + X_i), \tag{3.1}$$

where $\theta_{Y_i} \in [0, 1]$ and $q \in \mathbb{R}$.

Furthermore, Equation (3.1) is a linear combination in such a way that $\theta_{Y_i} = 1$ ($\theta_{Y_i} = 0$) lead to the Kijima type I (Kijima type II) model. Therefore, considering k alternatives (intervention types) for Y_j one has k new parameters to GRP, say $\theta = (\theta_1, \cdots, \theta_k)$ such that $\theta_{Y_i} \in \theta$, measuring the degrade between Kijima I and II models imposed to the system by each intervention type. In Equation (3.1), for $\theta_{Y_i} = 1$, the impact of the i^{th} intervention only operates on X_i, by qX_i. On the other hand, when $\theta_{Y_i} = 0$ the i^{th} intervention reflects on X_i and on the previous updated times between interventions, composing a geometric propagation of the quality of the i^{th} restoration on the overall system performance history. Traditionally, $\theta_{Y_i} = 1$ may characterize unintended interventions where the causes of the system stoppage are investigated in order to restore its continuation condition, only. On the other hand, $\theta_{Y_i} = 0$ might characterize intended interventions where eventual previous negligence of the maintenance crew on parts of the system can be inspected, identified, and then suppressed. However, depending on Y_j (*e.g.* whether the j^{th} intervention is planned or unplanned), the virtual age might be more or less affected by the j^{th} intervention.

From Definition 3.1.2, proposed here, the GRP virtual age function can fit the performance data set of the system in terms of both the times between interventions $\mathbf{X} = (X_1, X_2, ..., X_n)$ and the respective nature of such interventions $\mathbf{Y}$=$(Y_1, ..., Y_n)$ (*e.g.* whether planned or unplanned), besides the already known parameter q. To date, in the GRP literature, the vector $\mathbf{Y}$ is not taken into account. Based on the realization of stochastic processes, say $\{x_j, y_j\}_{j=1}^{i-1}$, we have

$$v_{i-1} = v((x_1, y_1), (x_2, y_2), ..., (x_{i-1}, y_{i-1}) \,|\, q).$$

In summary, it is suggested here that the level of restoration imposed to the system by each intervention might depends on the respective intervention type.

This scenario is not seen in literature, Krivtsov (2000) works with the GRP presenting a concept regarding virtual age functions considering the stochastic nature of the system restoration, but only considering the quality of intervention which is the traditional interpretations of virtual age that solely reflect the role that the parameter q plays on the system. This is the same kind of interpretation seen in Yañez et al. (2002), where models Kijima Type I and II are presented and their efficiency is presented separately. Specifically, they discuss the role of each model where Kijima Type I reacts better to interventions made only at the last time between interventions - situations where the intervention is unintended. This kind of model cannot capture with accuracy interventions made to affect the whole system. In this case, Kijima Type II is a model that expands the effect of repair through all history of the system - where planned interventions are made. Here, unintended interventions are not well modeled by Type II.

Here, by means of Equation (3.1), we present a mixed version where such degrade is possible. In this way, the lesser θ_{Y_j} the greater the impact of the interventions of type Y_j on the system performance. Thus, the proposed model also allows one to compare the quality of the existing intervention types.

Here, the probabilistic distribution used in GRP is the Weibull distribution, since its characteristics are widely applied with life time variables. Basically, the mathematical structure of probability functions remains the same as presented in Chapter 2, but the virtual age is now replaced by the mixed one proposed here. Thus, the MLE and random sampling generation are not modified.

However, an additional point is the interpretation of the WGRP parameters. The traditional meaning of parameters involving the WGRP presented in literature does not cover all possiblities that these parameters can assume and their impats on the system. A detailed study is presented in the next Section.

3.2 New meaning of parameters in WGRP

Let the hazard function of a WGRP aforementioned be presented as follows.

$$h_{T_i}(x + v_{i-1}|v_{i-1}, \alpha, \beta) = \frac{f_{T_1}(x + v_{i-1}|v_{i-1}, \alpha, \beta)}{1 - F_{T_1}(x + v_{i-1}|v_{i-1}, \alpha, \beta)} = \frac{\beta}{\alpha}\left(\frac{x + v_{i-1}}{\alpha}\right)^{\beta-1} \tag{3.2}$$

From Eq. (3.2) it is possible to analyze the meaning of the parameters according to the behaviour of the system in response to the interventions. Clearly, there are three situations: when $\beta > 1$, $\beta = 1$, and $\beta < 1$. For $\beta = 1$, we have the particular case of an Exponential distribution. When $\beta < 1$, the numerator and denominator will invert inside parenthesis and then, the greater the time (actual plus virtual ones) the lesser the hazard; thus the greater the q, the greater the virtual age, the better the interventions. On the other hand, if $\beta > 1$, the numerator will arise the hazard function as time turns higher, and then the greater the q the greater the hazard. In fact, this reasoning brings a more general interpretation for the meaning of the WGRP parameters in relation to the WGRP literature, mostly dedicated to the cases where $\beta > 1$.

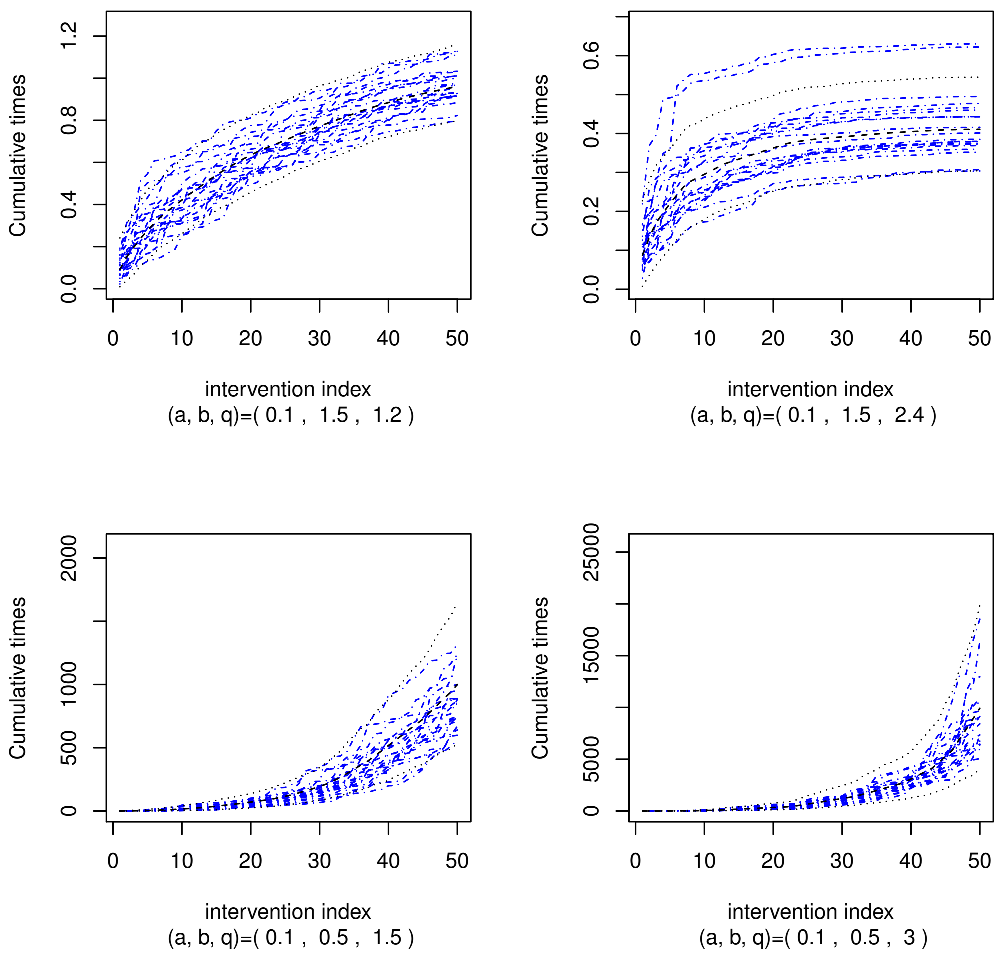

Figure 3: WGRP samples to explain relationship between values of $\beta > 1$, $\beta < 1$ and $q > 1$. The blue dashed lines represent WGRP samples. The black dashed line in bold represent the estimated WGRP mean model and the black point ones represent 95 percent confidence interval.

To reinforce these perspectives, Fig. 3 presents some simulations involving the relationship between β, q, and the stoppage times. It presents four situations to compare the behaviour of simulated times to understand how is the impact of the values of q

in two situations of β.In fact, as previously stated, if $\beta < 1$ then the greater the q the greater the level of system improvement imposed by the interventions, once times between interventions enlarge. In practice, this situation occurs when we have a system that is improving or learning with its stoppage events. This is common with software, artificial intelligence systems, and others that can learn at each step achieved. Thus, it is claimed that the literature traditionally interpret q for $\beta > 1$ (i.e. deteriorating systems) only, and neglects the meaning of q in the cases where $\beta < 1$.

Thus, it is more reasonable to think of β as being the main parameter for reflecting the system restoration pattern instead of q. In this way, the meaning of q depends on the behaviour of β. For instance, when $\beta = 1$, q becomes useless and the times between interventions are identically and exponentially distributed. In this situation, the system is considered without memory since the hazard function is constant through time (it only depends on α). Obviously, this case brings important discussions. In particular, it must reflect the situation where the intrinsic nature of the system as well as the performance of its maintenance crew are aligned in such a way that the system stays stable over the time. It would be seen as some kind of perfect balance, something possible though rare. These discussions arise the importance of a correct interpretation of the WGRP parameters, mainly of q. In fact, α and β can be understood similarly to the case of the Weibull distribution.

On the other hand, the mathematical challenges of WGRP has not been fully addressed. Guo et al. (2007) present an asymptomatic claim about the complexity of working mathematically with GRP. These authors state that the closed-form solution of the mean time between interventions is not available. In fact, there is little or even no work in WGRP literature that develops mathematical properties such as first moments. This problem is handled as follows.

3.3 First and second theoretical moments of WGRP

These moments might be important to promote preventive interventions analyses. In this way and without loss of generality the index of v_{i-1} is suppressed. The first moment is developed as follows:

$$\begin{aligned}
\mathbb{E}(X\,|\,\alpha,\beta,v) &= \int_0^\infty x\frac{\beta}{\alpha}\left(\frac{x+v}{\alpha}\right)^{\beta-1}\exp\left[\left(\frac{v}{\alpha}\right)^\beta-\left(\frac{x+v}{\alpha}\right)^\beta\right]dx\\
&= \exp\left[\left(\frac{v}{\alpha}\right)^\beta\right]\int_0^\infty x\frac{\beta}{\alpha}\left(\frac{x+v}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x+v}{\alpha}\right)^\beta\right]dx\\
&= \exp\left[\left(\frac{v}{\alpha}\right)^\beta\right]\left\{\int_{-v}^\infty x\frac{\beta}{\alpha}\left(\frac{x+v}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x+v}{\alpha}\right)^\beta\right]dx\right.\\
&\qquad\left.-\int_{-v}^0 x\frac{\beta}{\alpha}\left(\frac{x+v}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x+v}{\alpha}\right)^\beta\right]dx\right\}
\end{aligned}$$

Using some algebra and substitution, one has:

$$\mathbb{E}(X\,|\,\alpha,\beta,v) = \exp\left[\left(\frac{v}{\alpha}\right)^\beta\right]\left\{\alpha\Gamma\left(1+\frac{1}{\beta}\right)+\frac{\alpha}{\beta}\tilde{\Gamma}\left(\frac{1}{\beta},\left(\frac{v}{\alpha}\right)^\beta\right)-\frac{\alpha}{\beta}\tilde{\Gamma}\left(\frac{1}{\beta},0\right)\right\} \tag{3.3}$$

and $\mathbb{E}(X+v\,|\,\alpha,\beta,v)=\mathbb{E}(X\,|\,\alpha,\beta,v)+v$. Here, the incomplete Gamma function is given by $\tilde{\Gamma}(a,z)=\int_a^\infty e^{-t}t^{z-1}dt$ and must be approximated via numeric calculus.

In turn, the non-central second moment is given by:

$$\begin{aligned}
\mathbb{E}((X+v)^2\,|\,\alpha,\beta,v) &= \int_0^\infty (x+v)^2 f(x+v\,|\,\alpha,\beta,v)dx\\
&= \int_0^\infty x^2 f(x+v\,|\,\alpha,\beta,v)dx + 2v\int_0^\infty x f(x+v\,|\,\alpha,\beta,v)dx\\
&\qquad + v^2\int_0^\infty f(x+v\,|\,\alpha,\beta,v)dx\\
&= \exp\left[\left(\frac{v}{\alpha}\right)^\beta\right]\left\{\int_{-v}^\infty x^2 f(x+v\,|\,\alpha,\beta,v)dx-\int_{-v}^\infty x^2 f(x+v\,|\,\alpha,\beta,v)dx\right\}\\
&\qquad + 2v\mathbb{E}(x+v\,|\,\alpha,\beta,v)+v^2.
\end{aligned}$$

Finally, one has:

$$\begin{aligned}
\mathbb{E}((X+v)^2) = \exp\left[\left(\frac{v}{\alpha}\right)^\beta\right]&\left\{\alpha^2\Gamma\left(1+\frac{2}{\beta}\right)+\right.\\
&\left.+\,v^2+2\int_{-v}^0 x\exp\left[\left(\frac{x+v}{\alpha}\right)^\beta\right]dx\right\}+2v\cdot\mathbb{E}(X)+v^2.
\end{aligned} \tag{3.4}$$

These calculations conclude the first and second moments of X. Since these moments involve the exponential and the incomplete gamma functions. The presence of v in these functions may lead to computational problems as v increases. These are important results, since Doyen & Gaudoin (2006), Veber et al. (2008) and Moura et al. (2014) point that there is no presentation in literature about theoretical studies to WGRP.

Next Section discusses some numerical results where random sampling process is presented to the proposed model and an illustrative example of it.

3.4 Sampling process to expose first moment behaviour

We have already seen how to sample random WGRP series in Section 2.1.4. Through the use of Eq. (2.13), we can observe the behaviour of sample means and the first moment samples. By generating such sequence of WGRP random numbers, it is possible to notice the implications of the incomplete *Gamma* function.

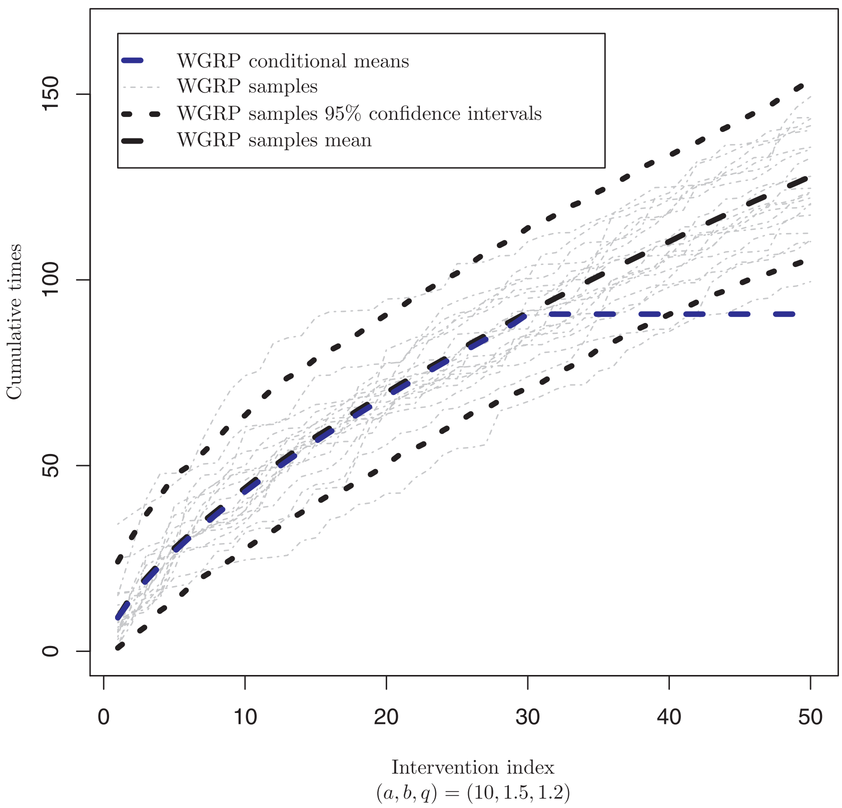

Figure 4: WGRP samples statistics and theoretical conditional expected values in an aging system ($\beta > 1$).

Considering an aging system where $\beta > 1$, $q = 1.2$, and $\theta_{y_i} = 1$ (Fig. 4), the times between interventions become closer gradually and the impact of the interventions is not so worth ($q = 1.2$). In other words, next intervention must be made in an earlier interval than the previous one and the intervention actions have contributed to such behaviour (otherwise, we would have q near 0). Furthermore, since $\beta > 1$, the power term in the first moment grows rapidly and the theoretical expected value quickly becomes intractable due to the complex math terms (see the blue dashed line).

4 Results

In this Chapter some results involving the adjustment of a WGRP model to three cases of literature are presented. Specifically, these cases suggest an increasing, constant and a decreasing hazard function. Then, analysis through Renewal Theory framework are presented separately.

First, the RP, NHPP, Kijima I, Kijima II, and proposed model were adjusted for each data set, via ML estimation, where the log-likelihood function for WGRP, Eq. (2.9), was optimized according to the simulated annealing algorithm provided by the GenSA package of the free-ware R software (R, 2009).

For the sake of comparison, the mean squared error (MSE) and log-likelihood (LL) metrics were computed for each model. Regarding the MSE, it was based on mean stoppage times, estimated from $200,000$ simulated samples from each model. In this context, it was considered the following space of possibilities for $\hat{\beta}$, $\hat{q}$, and $\hat{\theta}$: $(\hat{\beta}, \hat{q}, \hat{\theta}) \in [10^{-100}, 10] \times [-1.5, 1.5] \times [0, 1]^k$, where k is the number of alternatives for intervention.

4.1 Offshore facility data set

The first data set is from Langseth & Lindqvist (2005) where 84 stoppage times, regarding two intervention types (corrective (c) and preventive (p)) of a compressor system of an offshore facility is considered. Thus, the data set records two variables: times between maintenance actions and the respective types of maintenance. It is worthwhile to mention that the time until first intervention ($t_1 = 220$) was removed from the modelling study since it was considered an outlier. Thus, $X_0 = t_1$ in this case.

Firstly, the RP, NHPP, Kijima I, Kijima II, and proposed model were adjusted to the data set. Furthermore, the performance measures (LL and MSE) of each model were computed (see Table 1), via MLE. It is possible to conclude that the Kijima II model (the proposed one where $\theta = (0, 0)$) presents the best performance in terms of both LL and MSE.

Considering the proposed model, the maximum likelihood estimates for

α, β, q, and $\theta = (\theta_c, \theta_p)$ from the adopted data set (without t_1)

were $(\hat{\alpha}, \hat{\beta}, \hat{q}, (\hat{\theta}_c, \hat{\theta}_p))$=$(5.87, 0.95, 1.5, (0.0, 0.0))$. Thus, due to θ estimates, the proposed approach has been simplified to the Kijima II model.

To capture the inherent variability of the optimization process, the GenSA function was executed 30 times. However, the behaviour of the LL function related with the space of possibilities for parameters set was studied and it can be noticed a stabilized form of the function — it is clear through the graphic constructed for RP, NHPP, KI and KII models. To do so, values of LL were generated using values of β and q in their space to construct several graphics as shown bellow in each situation of Renewal Theory models.

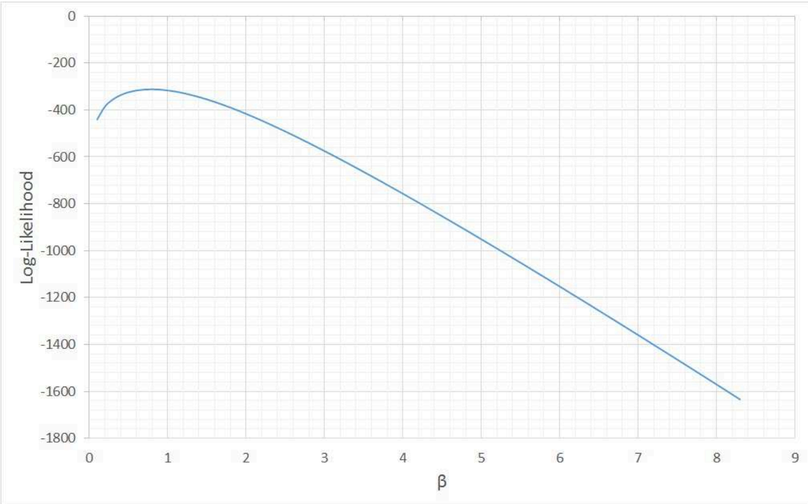

Figure 5: Curve of LL for RP model in Offshore database. The maximum is achieved in a single point considering values of β in a specific space.

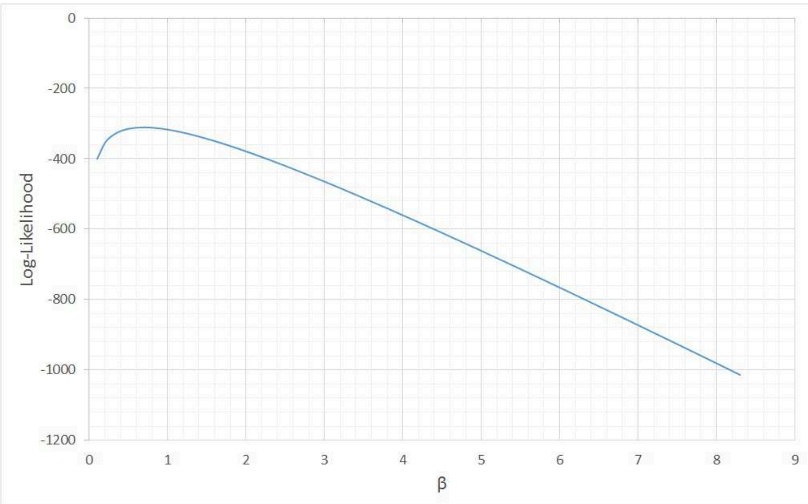

Figure 6: Curve of LL for NHPP model in Offshore database. The maximum is achieved in a single point considering values of β in a specific space.

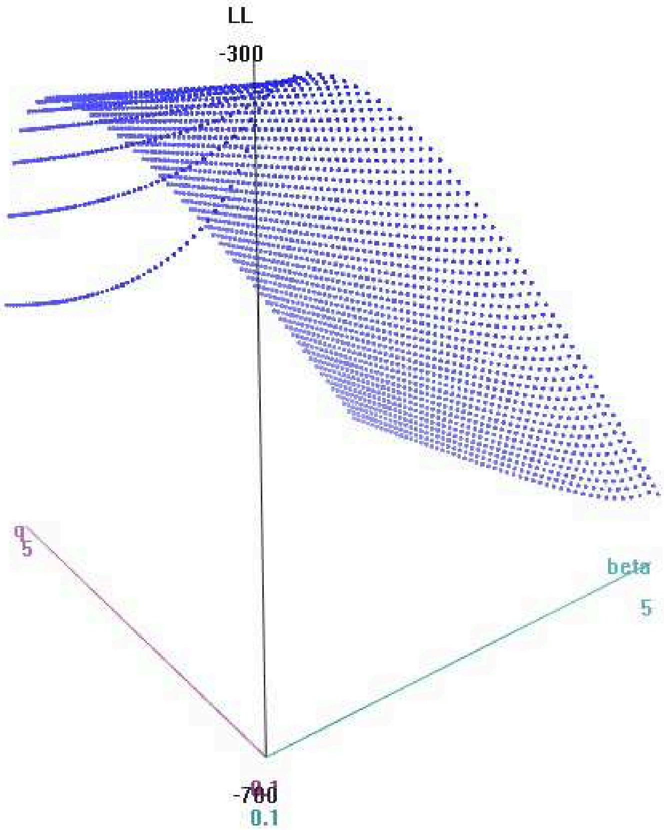

Figure 7: Curve of LL for KI model in Offshore database. The maximum is achieved in a single point considering values of β and q in a specific space.

Thus, descriptive measures does not indicate relevant variance in parameters set, once there is only one maximum value in LL function as seen in Fig. 5, Fig. 6, Fig. 7 and Fig. 8, respectively.

As $\hat{\beta} < 1$, it follows that the system is improving, in the sense that the greater the time the lower the hazard rate. It might reflect a burn-in period of the system, characterized by early failures attributable to defects in design, manufacturing, or construction (MODARRES et al., 1999). In turn, $(q, \theta_c, \theta_p) = (1.5, 0.0, 0.0)$ reveals that regardless the intervention type (whether corrective or preventive), its positive impact propagates through the entire system history with the maximum intensity, since $v_i = 1.5 \cdot (v_{i-1} + x_i)$ and it was assumed $q \leq 1.5$. It is scratched in Fig. 9 the observed stoppage time series and instances of the best fitted model.

In fact, it is exhibited in Fig. 9 the observed cumulative times $\boldsymbol{t}$=$(t_2, t_3, ..., t_{85})$, the respective WGRP sample points and 95% interval estimates as well as some of the series simulated from the fitted model. In this case, precise estimates from Eq. (3.3) were unavailable due to computational limitations. One can see that the fitted model has enveloped

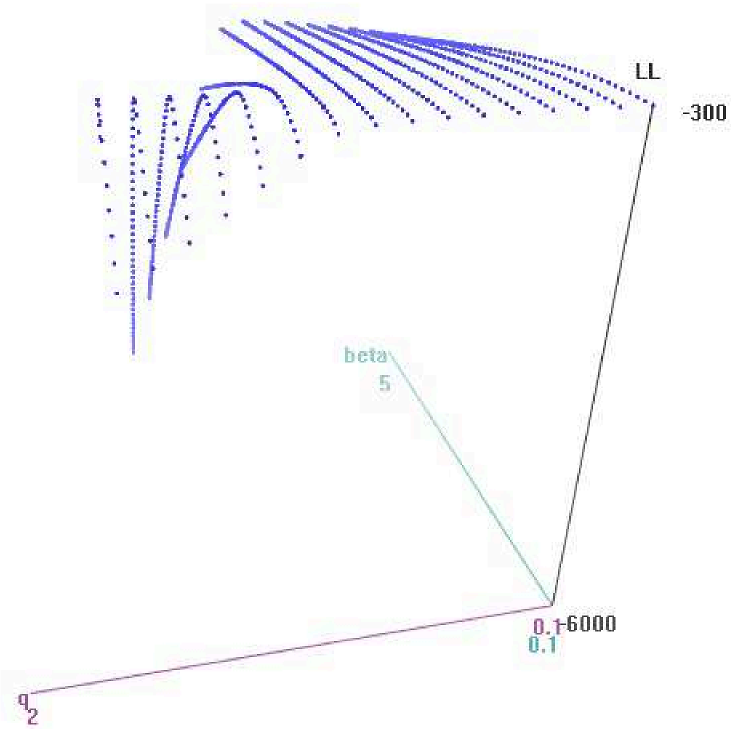

Figure 8: Curve of LL for KII model in Offshore database. The maximum is achieved in a single point considering values of β and q in a specific space.

Table 1: MLE parameters estimates, MSE and Log-Likelihood (LL) measures of WGRP models for the offshore facility dataset from Langseth & Lindqvist (2005)

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{q}$	$\hat{\theta}_c$	$\hat{\theta}_p$	MSE	LL
RP	14.39	0.79	*	*	*	49185.97	-312.27
NHPP	2.39	0.69	*	*	*	13991.01	-310.64
Kijima I	3.299	0.52	0.02	1	1	18542.23	-306.74
Kijima II	5.85	0.95	1.499	0	0	**	**
Mixed Kijima Model	5.85	0.95	1.499	0	0	3346.835	-306.63

* reflects the absence of the parameter in the model. ** means the same value found for both Kijima II and Mixed Kijima model.

the performance time series data set in such a way that the series is always in its 95% confidence interval estimates and the generated samples are similar to the real series, which

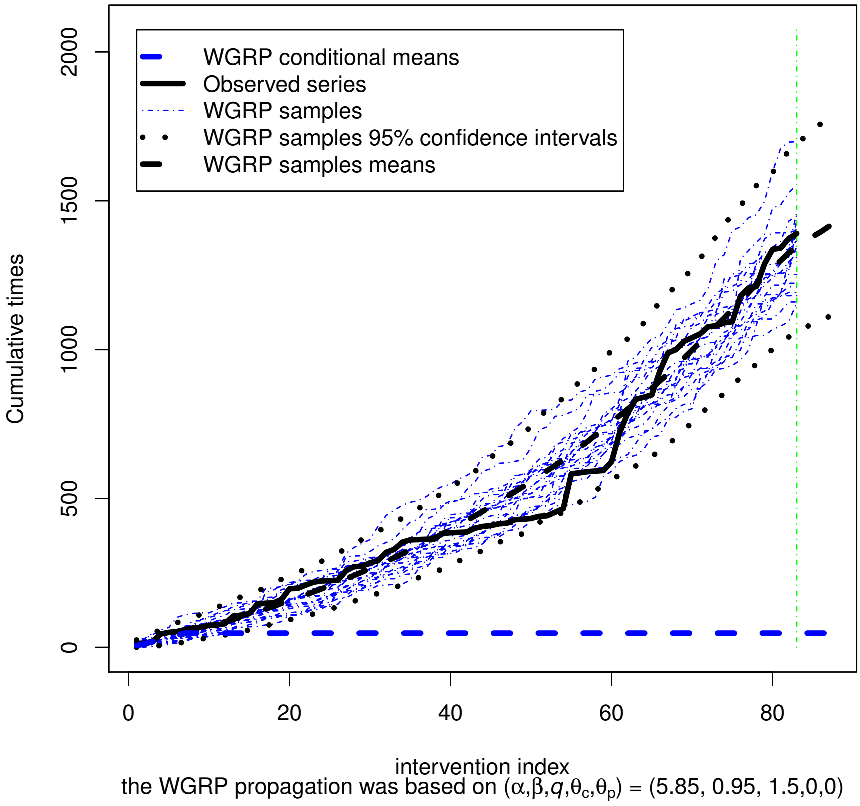

Figure 9: WGRP adjustment to the Langseth & Lindqvist (2005) Offshore oil data set without t_1.

indicates that the proposed WGRP model with $(\hat{\alpha}, \hat{\beta}, \hat{q}, (\hat{\theta}_c, \hat{\theta}_p))$=$(5.87, 0.95, 1.5, (0.0, 0.0))$ appears to be a suitable model for this situation.

Besides, it is also sketched in Fig. 9 the proposed preventive maintenances policy for the next 4 interventions, according to the adjusted model. Although $\boldsymbol{t}$ does not involve t_1, to sum up t_1 to the proposed preventive interventions policy, from the 85^{th} to the 88^{th} one, is a straightforward way for circumventing the problem.

It is presented in Table 2 the proposed instants for preventive interventions. Thus, in average, it is suggested, for instance, that the next preventive intervention should be performed in the instant 1657.7; however, it is inferred the system becomes unavailable at any instant in the interval $[1343.9, 2016.25]$, under a 95% confidence level. This provides some important conclusions, since the decision making can be based on these estimates, leading the maintenance crew to be under alert mode during these time interval.

Table 2: Policy for the next four preventive interventions of the system studied by Langseth & Lindqvist (2005).

intervention	85	86	87	88
2.5% quantile	1343.9	1399.76	1445.47	1513.81
mean	1657.7	1721.86	1801.9	1909.78
97.5% quantile	2016.25	2114.38	2261.27	2436.53

4.2 Windshield data set

The second data set involves 80 stoppage times regarding failure (say f) and service (say s) actions on a windshield system, from Murthy et al. (2004). Table 3 brings the MLE estimates of the alternative models and their respective log-likelihood and MSE metrics. The maximum likelihood estimates of α, β, q and (θ_f, θ_s) to the proposed model are $(\hat{\alpha}, \hat{\beta}, \hat{q}, (\hat{\theta}_f, \hat{\theta}_s))=$ $(0.06, 1.04, 1.5, (0.0, 0.0)$. Similarly to the previous case, the proposed model has been specified to the Kijima II approach and achieved the best results in terms of both LL maximization and MSE minimization. Also, graphics for LL function were produced to verify its behaviour.

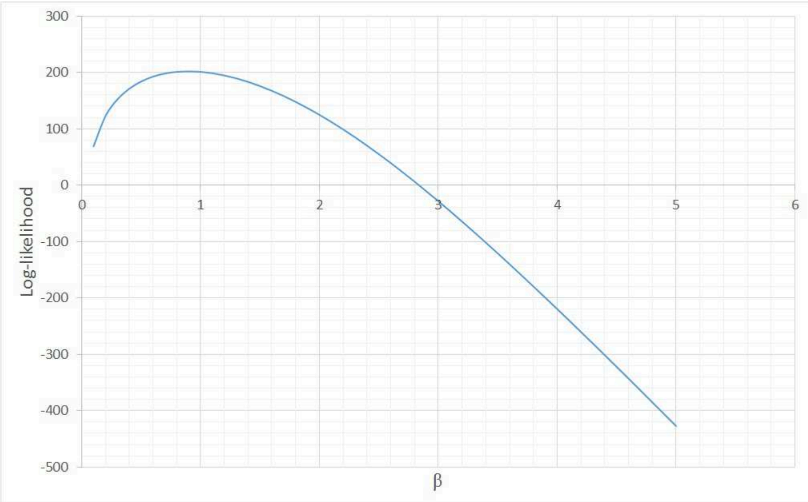

Figure 10: Curve of LL for RP model in Windshield database. The maximum is achieved in a single point considering values of β in a specific space.

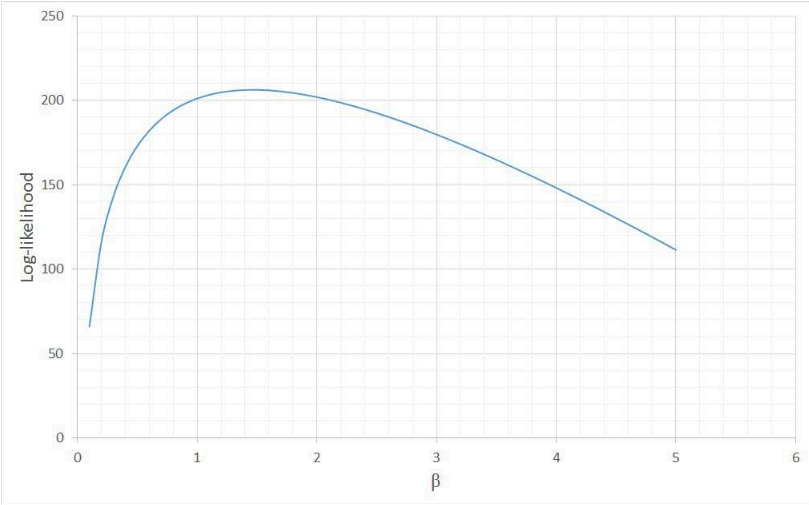

Figure 11: Curve of LL for NHPP model in Windshield database. The maximum is achieved in a single point considering values of β in a specific space.

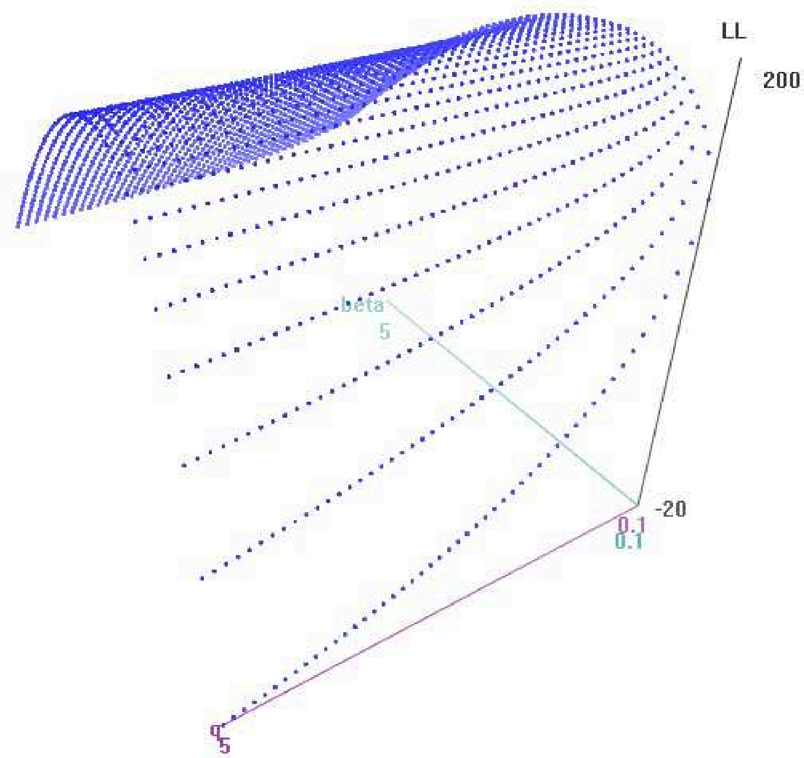

Figure 12: Curve of LL for KI model in Windshield database. The maximum is achieved in a single point considering values of β and q in a specific space.

This behaviour can be seen in Fig. 10, Fig. 11, Fig. 12, and Fig. 13.

As $\hat{\beta} > 1$, it is inferred the system is deteriorating; thus the greater the time the greater the hazard. It might represent a wear-out phase, mainly characterized by

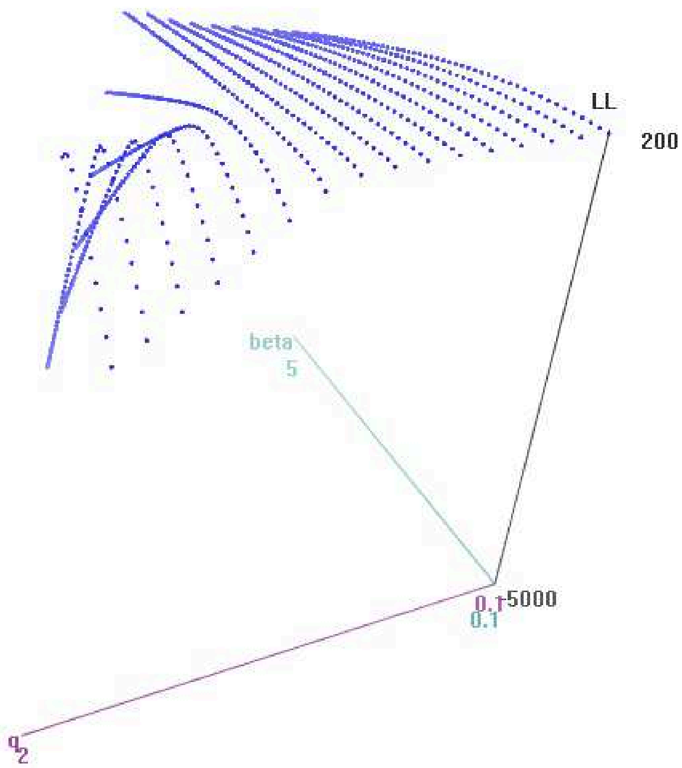

Figure 13: Curve of LL for KII model in Windshield database. The maximum is achieved in a single point considering values of β and q in a specific space.

Table 3: MLE parameters estimates of the WGRP models and respective MSE and LL measures for the Windshield data set from Murthy et al. (2004)

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{q}$	$\hat{\theta}_f$	$\hat{\theta}_s$	MSE	LL
RP	0.028	0.897	*	*	*	0.115	201.93
NHPP	0.119	1.465	*	*	*	0.0158	206.19
Kijima I	0.11	1.489	0.66	1	1	0.0151	206.22
Kijima II	0.0598	1.044	1.495	0	0	**	**
Mixed Kijima Model	0.0598	1.044	1.495	0	0	0.0053	207.99

* reflects the absence of this value in estimation process. ** means the same value found for both Kijima II and Mixed Kijima model.

complex aging phenomena, where the system deteriorates (e.g., due to accumulated fatigue) and is more vulnerable to outside shocks (MODARRES et al., 1999). In turn,

$(\hat{q}, \hat{\theta}_f, \hat{\theta}_s) = (1.5, 0.0, 0.0)$ indicates that regardless the intervention type (whether due to failure or service), its negative impact propagates through the entire system history with the maximum possible intensity, oppositely to the offshore facility case. This is caused due to the increasing hazard function ($\hat{\beta} > 1$), the deteriorating interventions ($\hat{q} > 1$), and the adoption of the Kijima II model ($\hat{\theta} = (0.0, 0.0)$). Thus, it is advised here the study of different ways to intervene, since the current ones seem to contribute to the system deterioration.

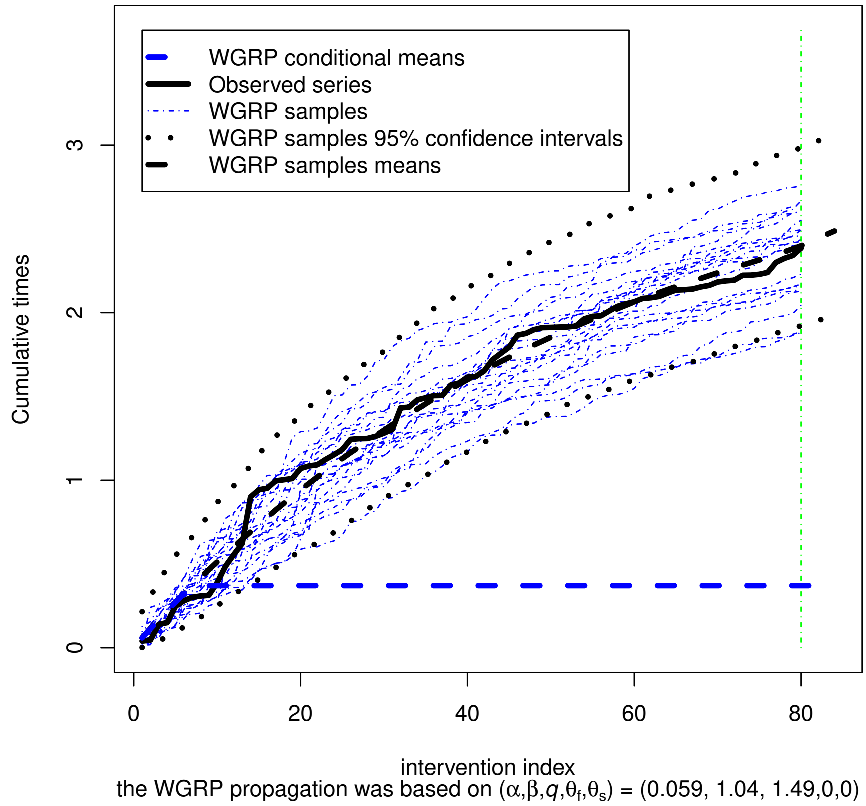

Figure 14: WGRP adjustment to the Murthy et al. (2004) Windshield data.

It is illustrated in Fig. 14 the observed cumulative times $\boldsymbol{t}$=$(t_2, t_3, ..., t_{80})$, the respective WGRP sample points and 95% interval estimates as well as some of the series simulated from the fitted model. Further, it is presented in Table 4 forecasts from this model for the next times between interventions.

Table 4: Preventive interventions policy for the windshield data set from Murthy et al. (2004) according the best WGRP model

intervention	81	82	83	84
2.5% quantile	1.93	1.95	1.97	2.01
mean	2.41	2.44	2.47	2.51
97.5% quantile	3	3.02	3.05	3.12

4.3 Transformers data set

This data set corresponds to 61 transformers stoppage events presented by Cristino (2008), the last data set considered in this work. Here, the intervention type is related to the complexity of the respective system: whether monophasic (the simplest case, represented here by letter m), or three-phase (say t). In Table 5, one can see the MLE's of the parameters for each model and the respective performance metrics. Following the same reasoning from other databases, this one presents a stable behaviour of their LL function.

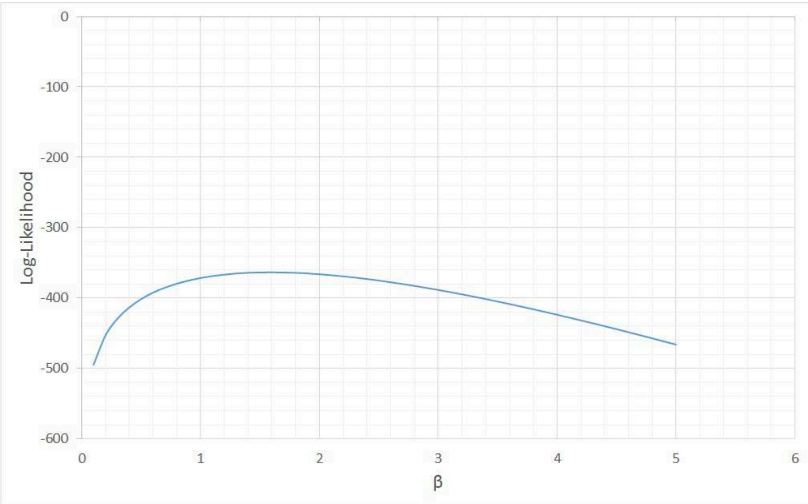

Figure 15: Curve of LL for RP model in Transformer database. The maximum is achieved in a single point considering values of β in a specific space.

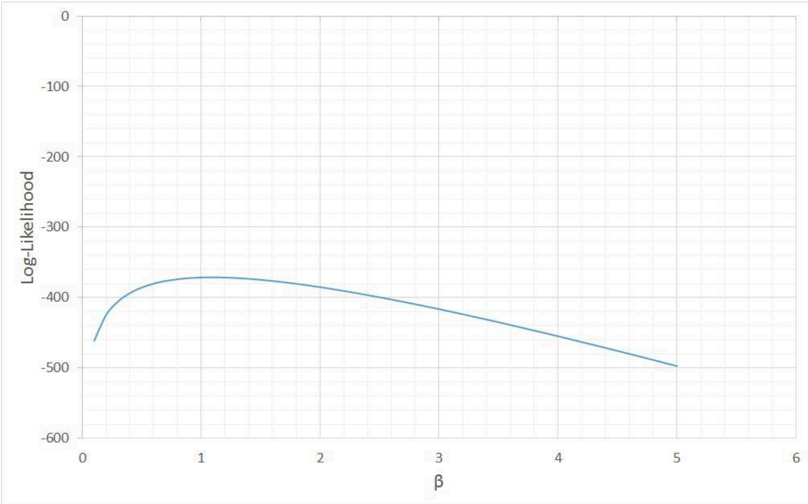

Figure 16: Curve of LL for NHPP model in Transformer database. The maximum is achieved in a single point considering values of β in a specific space.

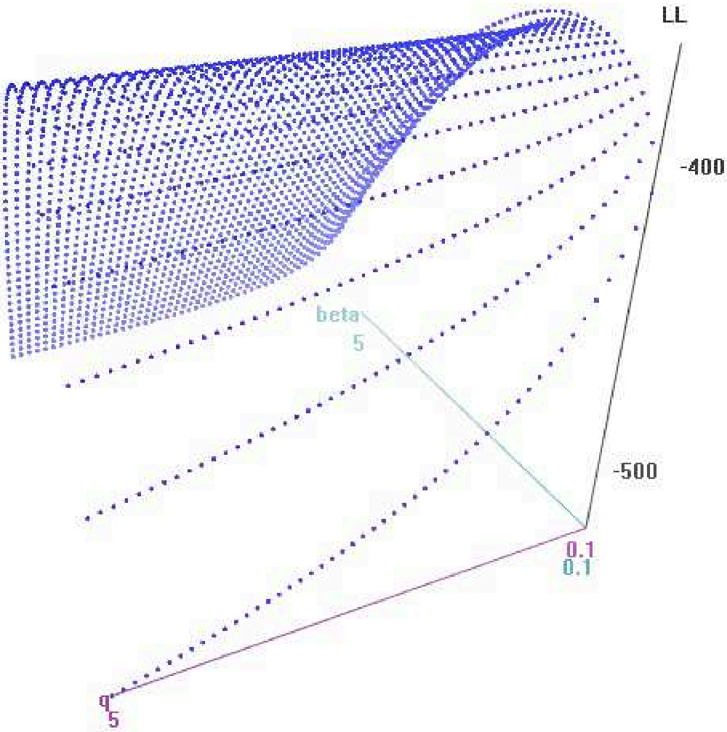

Figure 17: Curve of LL for KI model in Transformer database. The maximum is achieved in a single point considering values of β and q in a specific space.

As can be seen in Fig. 15, Fig. 16, Fig. 17, and Fig. 18, the behaviour of LL function for each model appears to be stable and present an achievable maximum.

Thus, differently from the previous cases, now the proposed approach suggests a

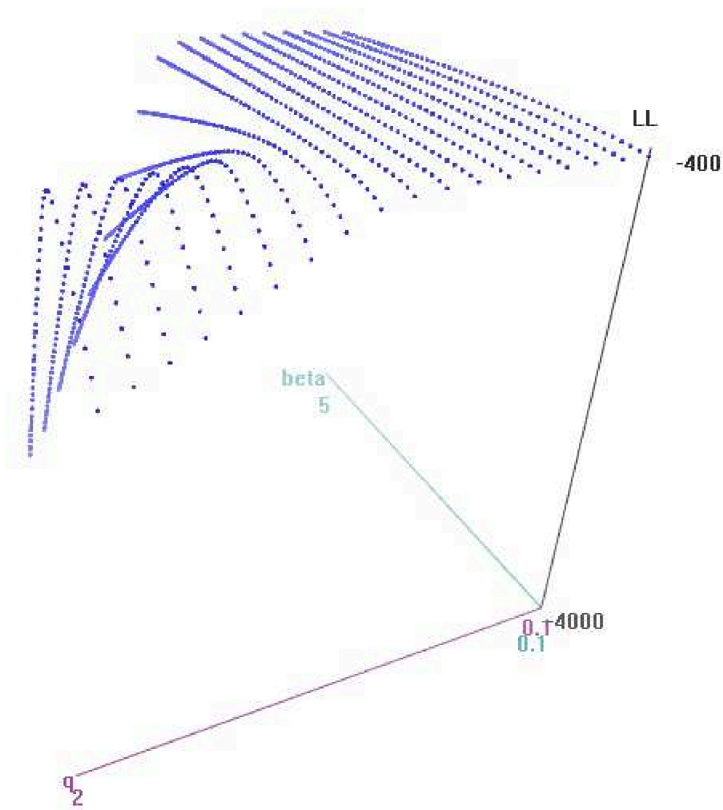

Figure 18: Curve of LL for KII model in Transformer database. The maximum is achieved in a single point considering values of β and q in a specific space.

Table 5: MLE estimates, MSE and LL measures of the WGRP models with respect to the Transformers dataset from Cristino (2008)

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{q}$	$\hat{\theta}_m$	$\hat{\theta}_t$	MSE	LL
RP	179.77	1.588	*	*	*	52929.59	-363.43
NHPP	227.14	1.088	*	*	*	23521.41	-371.26
Kijima I	210.34	1.91	0.005	1	1	21664.52	-361.78
Kijima II	273.11	2.336	0.3805	0	0	22094.55	-361.58
Mixed Kijima Model	282.53	2.519	0.2378	0.449	0.589	21436.53	-360.59

* means the absence of this value in estimation process.

degrade between the Kijima models for each type of intervention. The maximum likelihood estimates of α, β, q, and (θ_m, θ_t) are $(\hat{\alpha}, \hat{\beta}, \hat{q}, (\hat{\theta}_m, \hat{\theta}_t))$= $(282.53, 2.52, 0.24, (0.45, 0.59))$. As

$\hat{\beta} > 1$, we conclude that the system is deteriorating. Thus, the longer the time, the greater the hazard. However, as $\hat{q} \in (0, 1)$, the interventions have restored the system to an intermediate condition, between "as good as new"(where $q = 0$) and "as bad as old"(where $q = 1$).

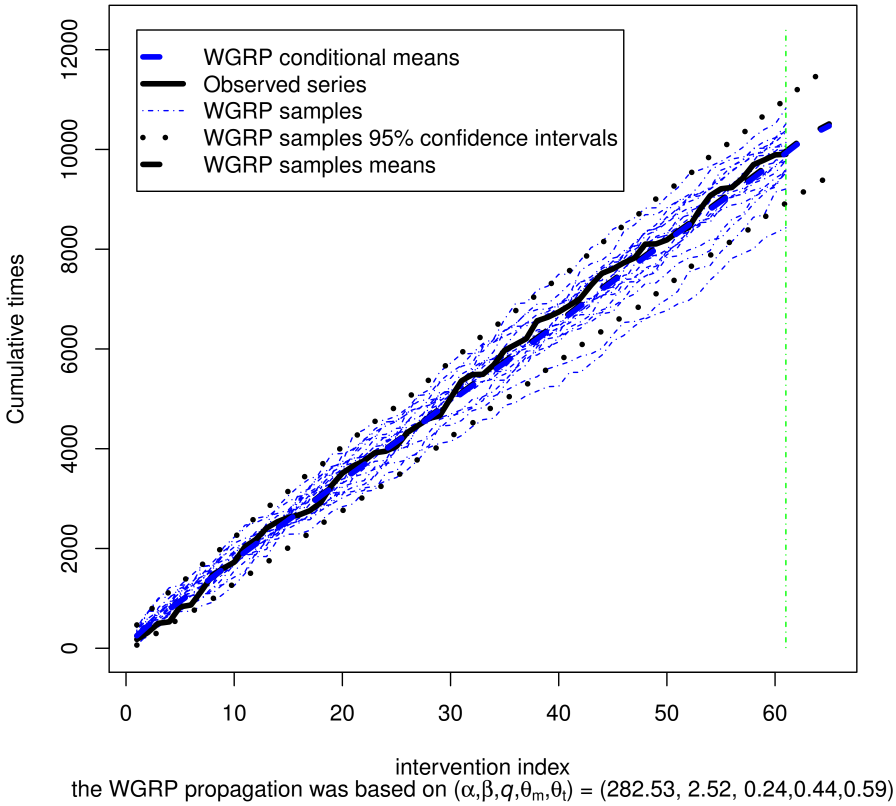

Figure 19: WGRP adjustment to the Cristino (2008) Transformers data.

In turn, as $\hat{\theta}_m < \hat{\theta}_t$ and the lesser the θ the greater the positive impact of the intervention (approaching a Kijima II model), one can conclude that interventions on the monophasic items (relative to θ_m) promote more restoration on the transformers system than the interventions on the three-phase ones (relative to θ_t). Such a phenomenon might result from the different levels of difficulty in performing interventions on monophasic and three-phase systems as well as from the skill of the maintenance team for dealing with these scenarios. Thus, the proposed model also allows to measure and compare the quality of the different types of intervention, providing support to decide about crew training and evaluation.

In Fig. 19, it is exhibited the observed cumulative times $\boldsymbol{t}$=$(t_2, t_3, ..., t_{61})$, the respective WGRP sample points and 95% interval estimates as well as some of the series

Table 6: Preventive interventions policy for the Transformers data from Cristino (2008), according to the best WGRP model

intervention	62	63	64	65
2.5% quantile	9064.05	9200.17	9336.77	9456.1
mean	10126.57	10273.83	10397.75	10510.76
97.5% quantile	11192.19	11334.12	11504.78	11602.52

simulated from the fitted model. From Fig. 19, one could infer that RP is adequate to the performance data set. However, it is clear from Table 5 that the RP is not among the best models. In fact, RP is the worst model in terms of MSE and the second worst with respect to LL. It allows one to conclude, from the best models, that there is a trade-off between the maintenance interventions and the deterioration underlying the system, leading to an apparent phase of constant-value hazard function. Such cases are characterized by random failures of the component; in this period, many mechanisms of failure due to complex underlying physical, chemical, or nuclear phenomena give rise to this approximately constant-value hazard function (MODARRES et al., 1999). Four forecasts from this model for the next times between interventions are presented in Table 6.

5 Conclusions

This work presents a new approach capable of acting in Renewal Theory framework. First, we present its functionality in Renewal Theory where its structure is a mixing process of Kijima models inside GRP. With this approach, we can analyze proportionally how each Kijima model is present in different interventions. This is made with the creation of two parameters inside the concept of virtual age model. It is important to notice that with this approach, we do not have to guess which Kijima model is used in databases. The approach can estimate proportionally the presence of those models in different types of interventions. Besides, the GRP model with this mixing is analyzed considering the Weibull distribution.

Furthermore, theoretical properties are developed and its consequences are studied through practical properties - random sampling processes. Through this, we present a new meaning of involved parameters, once literature usually does not treat cases where the value of shape parameter in WGRP is less than one. This new scenario is validated through numerical simulations and consequently brings new meanings to the rejuvenation parameter.

The approach is applied in real cases to verify its applicability in Renewal Theory comparing it with other models - RP, NHPP, Kijima I and II models. The choice of the best model is based on Mean Squared Error and Log-Likelihood measures. The approach can, at least, be equal to the other models and be the best when KI and KII are used in the database. These are important results, once now the decision maker does not need to guess which Kijima model is used in its problem, but we provide how much of each Kijima model is related in his intervention process.

In this way, the first two databases indicates that KII model is the best one- equally adjusted to the proposed approach. The last case brings a mixing of Kijima models and the used measures point to the new approach as the best model to be used.

After that, four preventive interventions are estimated based on ML estimators found. This is a tool to be used in polices of preventive interventions considering confidence intervals to each prediction.

We see potential to develop the same approach considering another probabilistic distributions, such as Gamma Distribution. Furthermore, a general hypothesis test can be developed to measure the adherence of the proposed approach. Besides, important regards about Competing Risks framework can be discussed in future works, such as a sophistication of the way to estimate who will occur first: censoring or undesired interventions - thinking in creating a dynamic estimation of their probabilities and therefore its role in the probabilistic behaviour of times between different interventions. Nowadays, this is a constant measure and could be modeled dependent on the time, for example. Also, in Renewal Theory framework, we can think in develop different models of virtual ages other than Kijima ones. The behaviour of q and its form is something that incites some analysis and further developments since the intervention crew can affect other aspects than real age of the system.

Bibliography

BAXTER, L. A. et al. On the tabulation of the renewal function. **Technometrics**, v. 24, p. 151–156, 1982.

BROWN, M.; PROSCHAN, F. Imperfect repair. **Journal of Applied Probability**, v. 20, n. 4, p. 851–859, 1983.

CRISTINO, C. T. **Risco e Confiabilidade sobre Estruturas Combinatórias: Uma Modelagem para Redes Elétricas**. Tese (Doutorado) — Mathematics Department of Universidade Federal de Pernambuco, 2008.

CROW, L. H. **Reliability analysis for complex, repairable systems**. [S.l.], 1975.

DAMASO, V. C.; GARCIA, P. A. d. A. Testing and preventive maintenance scheduling optimization for aging systems modeled by generalized renewal process. **Pesquisa Operacional**, scielo, v. 29, p. 563 – 576, 12 2009. ISSN 0101-7438. Disponível em: <http://www.scielo.br/scielo.php?script=sci_arttext&pid=S0101-74382009000300006&nrm=iso>.

DOYEN, L.; GAUDOIN, O. Imperfect maintenance in a generalized competing risks framework. **Journal of Applied Probability**, v. 43, p. 825–839, 2006.

FERREIRA, R. J.; FIRMINO, P. R. A.; CRISTINO, C. T. A mixed kijima model using the weibull-based generalized renewal processes. **PloS one**, Public Library of Science, v. 10, n. 7, p. e0133772, 2015.

GUO, H. R. et al. A new stochastic model for systems under general repairs. **Reliability, IEEE Transactions on**, IEEE, v. 56, n. 1, p. 40–49, 2007.

JACOPINO, A.; GROEN, F.; MOSLEH, A. Behavioural study of the general renewal process. In: IEEE. **Reliability and Maintainability, 2004 Annual Symposium-RAMS**. [S.l.], 2004. p. 237–242.

JACOPINO, A.; GROEN, F.; MOSLEH, A. Modelling imperfect inspection and maintenance in defence aviation through bayesian analysis of the kijima type i general renewal process (grp). In: IEEE. **Reliability and Maintainability Symposium, 2006. RAMS'06. Annual**. [S.l.], 2006. p. 470–475.

JAIN, M.; MAHESHWARI, S. Generalized renewal process (grp) for the analysis of software reliability growth model. **Asia-Pacific Journal of Operational Research (APJOR)**, v. 23, n. 02, p. 215–227, 2006. Disponível em: <http://ideas.repec.org/a/wsi/apjorx/v23y2006i02p215-227.html>.

JIMENEZ, P.; VILLALON, R. Generalized renewal process as an adaptive probabilistic model. In: **Transmission Distribution Conference and Exposition: Latin America, 2006. TDC '06. IEEE/PES**. [S.l.: s.n.], 2006. p. 1–6.

KIJIMA, M. Some results for repairable systems with general repair. **Journal of Applied Probability**, v. 26, n. 1, p. 89–102, 1989.

KIJIMA, M.; MORIMURA, H.; SUZUKI, Y. Periodical replacement problem without assuming minimal repair. **European Journal of Operational Research**, v. 37, n. 2, p. 194–203, November 1988. Disponível em: <http://ideas.repec.org/a/eee/ejores/v37y1988i2p194-203.html>.

KIJIMA, M.; SUMITA, M. A useful generalization of renewal theory: Counting processes governed by non-negative markovian increments. **Journal of Applied Probability**, v. 23, p. 71–88, 1986.

KRIVTSOV, V. **A Monte Carlo Approach to Modeling and Estimation of the Generalized Renewal Process in Repairable System Reliability Analysis**. Tese (Doutorado) — University of Maryland, College Park, 2000.

LANGSETH, H.; LINDQVIST, B. Competing risks for repairable systems: A data study. **Journal of Planning and Inference**, v. 136, n. 5, p. 1687–1700, 2005.

MODARRES, M.; KAMINSKIZ, M.; KRIVSTOV, V. **Realiability Engineering and Risk Analysis: A Practical Guide**. Marcel Dekker Incorporated, 1999. (Plastics Engineering). ISBN 9780824720001. Disponível em: <http://books.google.com.br/books?id=IZ5VKc-Y4_4C>.

MOURA, M. das C. et al. A competing risk model for dependent and imperfect condition–based preventive and corrective maintenances. **Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability**, SAGE Publications, v. 228, n. 6, p. 590–605, 2014.

MURTHY, D. P.; XIE, M.; JIANG, R. **Weibull models**. [S.l.]: John Wiley & Sons, 2004. v. 505.

R, D. C. T. **R: A language and environment for statistical computing**. Vienna, Austria: R Foundation for Statistical Computing, 2009. 409 p. ISBN 3-900051-07-0.

RIGDON, S. E.; BASU, A. P. **Statistical methods for the reliability of repairable systems**. [S.l.]: Wiley New York, 2000.

ROSS, S. **Simulation**. [S.l.]: Academic Press, 1997. (Statistical Modeling and Decision Science). ISBN 9780125984102.

ROSS, S. M. **Introduction to Probability Models**. [S.l.]: Elsevier Science, 2006. ISBN 9780123756879.

VEBER, B.; NAGODE, M.; FAJDIGA, M. Generalized renewal process for repairable systems based on finite weibull mixture. **Reliability Engineering & System Safety**, v. 93, n. 10, p. 1461 – 1472, 2008. ISSN 0951-8320. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0951832007002499>.

WEIBULL, W. A statistical distribution function of wide applicability. **ASME Journal of Applied Mechanics**, v. 51, p. 293–297, 1951.

YAÑEZ, M.; JOGLAR, F.; MODARRES, M. Generalized renewal process for analysis of repairable systems with limited failure experience. **Reliability Engineering & System Safety**, v. 77, n. 2, p. 167 – 180, 2002. ISSN 0951-8320. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0951832002000443>.

Appendix

APPENDIX A – Algorithm used to Estimation Process

Step 1: Choose the database to be evaluated;

Step 2: Set initial values of parameters set — $\hat{\beta}$, $\hat{q}$, and $\hat{\theta}$: $(\hat{\beta}, \hat{q}, \hat{\theta})$
$\in [10^{-100}, 10] \times [-1.5, 1.5] \times [0, 1]^k$;

Step 3: Set control conditions to optimization GenSA function — maxit = 10000, nb.stop.improvement = 50, smooth = TRUE, max.call = 300000, max.time = 60*10, temperature = 15000, verbose = FALSE (parameters from the function);

Step 4: Call GenSA to optimize the following set — par=as.vector(par), lower = lower, upper = upper, fn = objectiveFunction, control=control (parameters used in GenSA);

Step 5: Capture the set of maximum parameters and LL values to each model —- RP, NHPP, KI, KII, and MK model;

Step 6: Calculate the MSE for each model and construct a table;

Step 7: Construct the graphic for the best fit in each database;

Step 8: Estimate the next four interventions in each database.

APPENDIX B – Databases

Offshore Database					
TBI	**IT**	**TBI**	**IT**	**TBI**	**IT**
13	PM	12	PM	2	CM
1	PM	7	PM	30	CM
6	PM	28	CM	97	CM
25	PM	10	PM	65	CM
5	PM	24	CM	47	PM
3	PM	8	CM	7	PM
6	CM	1	CM	8	PM
6	PM	1	PM	80	CM
2	CM	1	PM	61	CM
7	PM	19	CM	11	CM
1	PM	2	PM	28	PM
5	PM	1	PM	12	PM
25	CM	1	PM	13	CM
3	CM	13	PM	24	PM
5	CM	6	PM	3	PM
32	PM	3	PM	10	PM
3	PM	6	PM	4	CM
1	PM	2	PM	85	CM
12	PM	12	PM	28	PM
36	PM	1	PM	5	PM
1	PM	3	PM	76	CM
11	CM	7	PM	49	PM
10	CM	2	PM	4	PM
4	CM	12	CM	32	PM
1	PM	12	CM	17	PM
1	PM	117	CM		
32	PM	3	CM		
14	PM	4	CM		
1	PM	2	CM		

TBI: Times Between Intervention; IT: Intervention Types; PM: Preventive Maintenance; CM: Corrective Maintenance

Windshield Database					
TBI	**IT**	**TBI**	**IT**	**TBI**	**IT**
0.04	Failure	0.013	Service	0.029	Failure
0.006	Service	0.019	Failure	0.028	Failure
0.094	Service	0.022	Failure	0.015	Service
0.01	Service	0.129	Failure	0.012	Service
0.098	Service	0.004	Service	0.02	Failure
0.032	Service	0.044	Failure	0.004	Failure
0.021	Failure	0.012	Service	0.008	Failure
0.008	Failure	0.013	Failure	0.02	Service
0.004	Service	0.001	Failure	0.018	Failure
0.076	Service	0.062	Failure	0.002	Service
0.098	Service	0.012	Service	0.004	Service
0.07	Failure	0.035	Failure	0.013	Failure
0.065	Service	0.004	Failure	0.009	Service
0.278	Service	0.033	Failure	0.02	Service
0.043	Failure	0.067	Service	0.007	Failure
0.009	Service	0.038	Failure	0.004	Failure
0.044	Service	0.037	Service	0.029	Failure
0.007	Service	0.072	Failure	0.001	Failure
0.007	Service	0.01	Failure	0.005	Failure
0.06	Failure	0.023	Failure	0.011	Service
0.015	Service	0.012	Failure	0.06	Failure
0.007	Service	0.001	Failure	0.024	Failure
0.032	Failure	0.002	Failure	0.017	Service
0.028	Service	0.001	Service	0.044	Failure
0.031	Service	0.005	Service		
0.061	Service	0.043	Service		
0.004	Failure	0.015	Service		
0.001	Service	0.003	Failure		

TBI: Times between Interventions; IT: Intervention Types

Transformer Database					
TBI	**IT**	**TBI**	**IT**	**TBI**	**IT**
178	three-phase	344	three-phase	12	monophase
135	three-phase	128	three-phase		
187	three-phase	6	three-phase		
29	three-phase	191	three-phase		
302	three-phase	291	monophase		
36	monophase	115	three-phase		
288	monophase	117	three-phase		
314	monophase	359	monophase		
135	monophase	72	monophase		
121	monophase	104	monophase		
314	three-phase	131	monophase		
141	monophase	142	monophase		
230	three-phase	268	monophase		
127	three-phase	234	three-phase		
85	three-phase	87	monophase		
51	monophase	123	monophase		
86	monophase	96	three-phase		
162	three-phase	280	three-phase		
342	three-phase	4	monophase		
247	three-phase	81	three-phase		
144	three-phase	160	three-phase		
102	three-phase	83	monophase		
163	three-phase	350	three-phase		
31	three-phase	294	three-phase		
97	three-phase	137	three-phase		
277	monophase	26	three-phase		
142	monophase	187	monophase		
141	monophase	272	monophase		
54	three-phase	93	monophase		
353	three-phase	102	monophase		

TBI: Times between Intervention; IT: Intervention Types